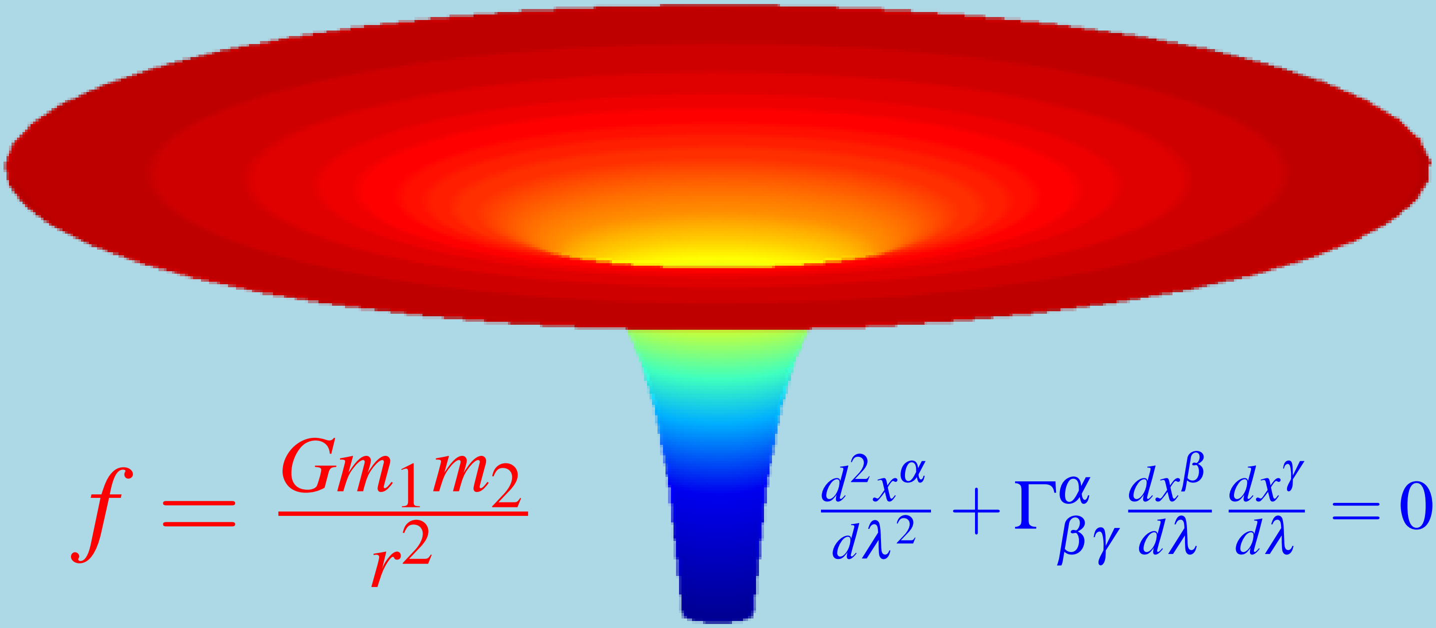
Solutions of Exercises of General Relativity Simplified & Assessed

$$(d\sigma)^2 = \left(1 - \frac{2GM}{c^2 r}\right)(cdt)^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1}(dr)^2 - r^2 (d\theta)^2 - r^2 \sin^2\theta\,(d\phi)^2$$



$$f = \frac{Gm_1 m_2}{r^2}$$

$$\frac{d^2 x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0$$

$$\nabla^2\Phi = 4\pi G\rho$$

$$G_{\mu\nu} = \kappa T_{\mu\nu}$$

Taha Sochi

Preface

This book contains the detailed solutions of all the 606 exercises of my book: General Relativity Simplified & Assessed. The exercises and their solutions represent an integral part of the original book as they fill many gaps and provide essential extensions and elaborations. Therefore, the readers are urged to give adequate attention to this book (as to the original book) so that they acquire a more comprehensive view and better understanding of the contents of the book and its subject.
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London, July 2020

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Nomenclature

In the following list, we define the common symbols, notations and abbreviations which are used in the book as a quick reference for the reader. The list generally excludes what is used locally and casually.

∇	nabla differential operator
∇f	gradient of scalar f
$\nabla \cdot \mathbf{A}$	divergence of vector $\mathbf{A}$
$\nabla \times \mathbf{A}$	curl of vector $\mathbf{A}$
∇^2	Laplacian operator
$\Box$	nabla differential 4-operator
$\Box^2$	d'Alembertian 4-operator
, (subscript)	partial derivative with respect to the following index(es)
; (subscript)	covariant derivative with respect to the following index(es)
nD	n-dimensional
$\delta/\delta\lambda$, $\delta^2/\delta\lambda^2$	absolute derivative operators with respect to λ
∂_i, ∂^i	partial differential operators with respect to i^{th} coordinate
$[ij, k]$, $[\mu\nu, \omega]$	Christoffel symbol of the 1^{st} kind in 3D (or nD) space, in 4D spacetime
a	mean distance (in planetary motion)
a	semi-major axis of ellipse or transverse semi-axis of hyperbola
$\mathbf{a}$	acceleration vector in 3D space
A	area
A	constant in the orbital equations
a, Δ, ρ^2	parameters in Kerr metric
b	semi-minor axis of ellipse or conjugate semi-axis of hyperbola
B	constant representing magnitude of angular momentum per mass or twice areal speed
B2, B3, B4	author books (see References)
B2X, B3X, B4X	author exercise books (see References)
c	the characteristic speed of light in vacuum (= 299792458 m/s)
ct, r, θ, ϕ	Schwarzschild coordinates, Boyer-Lindquist coordinates
C_s	constant equal to Schwarzschild radius
d	distance
$d\mathbf{r}$	differential of position vector
ds, $d\sigma$	infinitesimal line element in 3D (or nD) space, in 4D spacetime
$\mathbf{D}$, D^μ	displacement (or deviation) 4-vector
$\mathrm{diag}\,[\cdots]$	diagonal matrix with the embraced diagonal elements $\cdots$
e	eccentricity of ellipse
$\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi$	basis vectors of normalized spherical coordinate system
E, E_0	energy, rest energy
$\mathbf{E}_i$, $\mathbf{E}^i$	i^{th} covariant and contravariant basis vectors
Eq., Eqs.	Equation, Equations
ESA	European Space Agency
f	magnitude of force in 3D space
F	dimensionless factor in Schwarzschild metric ($= 1 - [R_S/r]$)
$\mathbf{f}$	force vector in 3D space
f_t	tidal force
g	magnitude of gravitational acceleration or gravitational force field
g	determinant of covariant metric tensor
$\mathbf{g}$	gravitational acceleration or gravitational force field
G	the gravitational constant ($\simeq 6.674 \times 10^{-11}$ N m^2/kg^2)
G	trace of Einstein tensor

$\mathbf{G}, G_{\mu\nu}, G^{\mu\nu}, G^{\mu}_{\nu}$	Einstein tensor
g_{ij}, g^{ij}, g^{i}_{j}	metric tensor of 3D (or nD) space
$g_{\mu\nu}$, $g^{\mu\nu}$, g^{μ}_{ν}	metric tensor of 4D spacetime
GPS	Global Positioning System
h	Planck constant ($\simeq 6.62607 \times 10^{-34}$ J s)
$\hbar$	reduced Planck constant ($= h/2\pi$)
H, K	functions of r in Schwarzschild metric
iff	if and only if
J	magnitude of angular momentum
k	Boltzmann constant ($\simeq 1.38065 \times 10^{-23}$ J/K)
L	length
LAGEOS	LAser GEOdynamics Satellites
LIGO	Laser Interferometer Gravitational-wave Observatory
LISA	Laser Interferometer Space Antenna
m, M	mass
m_g, m_i	gravitational mass, inertial mass
n	number of dimensions of space
n	number of particles inside a given volume
N	number of components or elements
N, N_0	number density, proper number density
N_c	number of planetary revolutions per century
$\mathbf{n}$, n_i	normal unit vector to surface
NASA	National Aeronautics and Space Administration
$\mathbf{p}$, $\mathbf{P}$	momentum vector in 3D space, in 4D spacetime
Q	electric charge
r	radius, radial distance, radial coordinate
$\mathbf{r}$	position vector [e.g. (x, y, z) or (x^1, x^2, x^3) or $(x^1, \cdots, x^n)$]
r_0, r_1	perihelion distance, aphelion distance
r, ϕ	plane polar coordinates
r, θ, ϕ	spherical coordinates of 3D space
R	radius, radial distance
R	Ricci curvature scalar
R_{ij}, R^{i}_{j}	Ricci curvature tensor of the 1^{st} and 2^{nd} kind in 3D (or nD) space
$R_{\mu\nu}$, $R^{\mu}{}_{\nu}$	Ricci curvature tensor of the 1^{st} and 2^{nd} kind in 4D spacetime
R_{ijkl}, $R^{i}{}_{jkl}$	Riemann-Christoffel curvature tensor of the 1^{st} and 2^{nd} kind in 3D (or nD) space
$R_{\mu\nu\rho\omega}$, $R^{\mu}{}_{\nu\rho\omega}$	Riemann-Christoffel curvature tensor of the 1^{st} and 2^{nd} kind in 4D spacetime
R_S	Schwarzschild radius
s	space interval (or length of arc)
$S_{\mu\nu}$, $S^{\mu\nu}$, $S^{\mu}{}_{\nu}$	electromagnetic field strength tensor
SAO	Smithsonian Astrophysical Observatory
t	time, temporal coordinate
T	periodic time
T	temperature
T	trace of energy-momentum tensor
T (superscript)	transposition
T_c	time of a century
$\mathbf{T}$, T^i, T_i	unit tangent vector
$\mathbf{T}, T_{\mu\nu}, T^{\mu\nu}, T^{\mu}{}_{\nu}$	energy-momentum tensor
u	$1/r$
u	speed
$\mathbf{u}$	geodesic unit vector
$\mathbf{u}$	velocity vector in 3D space

$\mathbf{U}$, U^{μ}	velocity vector in 4D spacetime
v	speed
$\mathbf{v}$	velocity vector in 3D space
V	volume
v_{es}	escape speed
V_{GR}, V_N	general relativistic effective potential, Newtonian effective potential
VLBI	Very Long Baseline Interferometry
$\mathbf{x}$	position vector in 4D spacetime, i.e. (x^0, x^1, x^2, x^3)
x_i	i^{th} Cartesian coordinate (or covariant form of general coordinates)
x^i, x^{μ}	general coordinates of 3D (or nD) space, of 4D spacetime
x, y, z	coordinates of 3D space (usually orthonormal Cartesian)
α	angle of precession per revolution (in geodetic effect)
γ	Lorentz factor
Γ^k_{ij}, $\Gamma^{\omega}_{\mu\nu}$	Christoffel symbol of the 2^{nd} kind in 3D (or nD) space, in 4D spacetime
$\delta_{ij}, \delta^{ij}, \delta^i_j$	Kronecker delta tensor in 3D (or nD) space
$\delta_{\mu\nu}, \delta^{\mu\nu}, \delta^{\mu}_{\nu}$	Kronecker delta tensor in 4D spacetime
$\delta\phi$	angle of deflection of light
$\delta\phi$	extra precession of perihelion per revolution
$\delta\phi_c$	extra precession of perihelion per century
Δ	finite change
ε	trace of perturbation tensor
ε_0	the permittivity of free space
$\varepsilon_{\mu\nu}$, $\varepsilon^{\mu\nu}$, ε^{μ}_{ν}	perturbation tensor
$\eta_{\mu\nu}, \eta^{\mu\nu}$	Minkowski metric tensor
κ	coefficient of the Field Equation ($= -8\pi G c^{-4}$)
κ_g, κ_n	geodesic and normal curvatures
$\mathbf{K}$	curvature vector
$\mathbf{K}_g$, $\mathbf{K}_n$	geodesic and normal components of curvature vector
λ	natural (or affine) parameter of curve
λ	wavelength
Λ	cosmological constant
μ_0	the permeability of free space
ν	frequency
ρ, ρ_0	density, proper density
ρ, ϕ, z	cylindrical coordinates of 3D space
σ	spacetime interval
σ	Stefan-Boltzmann constant ($\simeq 5.67037 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$)
τ	proper time
Φ	gravitational potential
$\chi^{\mu\nu}$	perturbation tensor
ω	angular speed

Chapter 1
Preliminaries

1.1 Synopsis about Gravity and General Relativity

1. Explain why gravity, unlike electric force, is the dominant force on large scale although they have identical dependency on distance since both are subject to the same inverse square relation (classically at least).
   Answer: The dominance of gravity arises from its dependency on mass (which is a general attribute of matter) regardless of any specific attribute of matter like charge (which is a restricted attribute of matter). Although electric force is identical to gravity in its inverse square dependency on distance, its influence diminishes faster than gravity due to its dependency on charged matter because matter becomes increasingly neutral on large scale and hence the electric force fades away faster than the gravitational force. In brief, gravity is an attribute of matter (any matter) while electric force is an attribute of charged matter (i.e. a special type of matter) and hence its influence is more restricted than gravity.
2. Compare mass and electric charge in their relation to gravity and electromagnetism.
   Answer: We may note two main things:
   • It seems that in our Universe we have only one type of mass (say positive mass) but two types of electric charge (i.e. positive and negative).
   • While similar charges repel each other (and dissimilar charges attract each other) "similar" masses attract each other.
3. Compare, briefly, between gravity in classical physics and gravity in general relativity.
   Answer: We note the following:
   • Classical gravity is a force while general relativistic gravity is an effect of the geometry of spacetime.
   • Classical gravity is a spatial phenomenon and hence it has no inherent temporal dependency while general relativistic gravity is a tempo-spatial phenomenon and hence it has inherent temporal dependency as well as spatial dependency.
   • Classical gravity depends on the distribution of mass while general relativistic gravity depends on the distribution of mass and energy (as well as their movement and stress).
   Note: there is a connection between the second and third points. This is because energy (divided by c) is the zeroth component of the momentum 4-vector according to the tensor formulation of Lorentz mechanics (see § 2.14 and refer to our book "The Mechanics of Lorentz Transformations") and hence it corresponds to time. Therefore, the temporal dependency of gravity and the inclusion of energy as a source of gravity originate from the same physical principle.
4. Outline the need for general relativity and the factors that contributed to the emergence of this theory.
   Answer: One factor that contributed to the emergence of general relativity is the emergence of Lorentz mechanics and the desire to extend this new mechanics to include gravity following the success of Lorentz mechanics in unifying mechanics and electromagnetism under the Lorentz transformations. Another factor is the need to address some limitations of the classical Newtonian gravitation theory, such as the lack of temporal dependency (which leads to violation of the speed limit imposed by special relativity) and the failure of classical theory in certain theoretical predictions such as the perihelion precession of Mercury.
5. According to general relativity, what is the characteristic feature of the world line followed by an object under the influence of gravity alone (and hence it is in a state of free fall)? Consider in your answer massless objects (such as light) as well as massive objects.
   Answer: According to general relativity, the world line followed by an object that is not influenced by any force other than gravity (which is not a force according to general relativity) is characterized by being a geodesic curve in the spacetime. This equally applies to massive and massless objects.

6. Establish a connection between special relativity and general relativity with regard to the underlying spaces of these theories and the motion of physical objects in these spaces.
   Answer: Special relativity is based on a flat spacetime that is void of matter and energy and hence the motion in such a spacetime is free and follows straight geodesic paths (i.e. world lines) in the absence of other forces (gravity is already assumed absent due to the absence of matter and energy). By the equivalence principle, the theory is generalized by including gravity where gravity is modeled as geometric curvature in spacetime using the concepts and techniques of differential geometry. Accordingly, the motion of physical objects in the spacetime of general relativity in the absence of other forces also becomes a free motion but it follows curved geodesic paths due to the curvature of the underlying spacetime.
7. Outline the main criticisms that faced Newtonian gravity and contributed to the emergence of general relativity.
   Answer: The Newtonian theory of gravity was criticized for its failure to provide a precise prediction for the advance of perihelion of Mercury (see § 8.1). The theory was also criticized for being purely spatial with no temporal dependency and hence it predicts action at a distance, i.e. instantaneous interaction between two spatially separated objects which implies infinite speed of communication and exchange of signals between physical objects in violation of the special relativistic restriction that no speed can exceed c. The theory was also criticized for its incompatibility with the framework of special relativity since gravity (unlike mechanics and electromagnetism) does not transform invariantly under the Lorentz transformations.
8. Briefly discuss the implications of the fact that general relativity is a geometric theory.
   Answer: The main implication is that it may not capture the essence and express the spirit of the physics behind the physical phenomena (i.e. gravity and its effects) and hence the phenomena will be described by their symptoms rather than by their essence. Ideally a physical phenomenon like gravity should be described by physical (rather than geometrical) concepts and techniques to fully represent the physics behind the phenomenon. In fact, general relativity may be compared to quantum mechanics in this regard as both theories potentially provide correct but rather sightless rules about the behavior of Nature. In other words, they may be formally correct but they may not be ideal or perfect especially from an epistemological perspective. In brief, due to its geometric nature general relativity is not an intuitive and natural physical theory (see § 10.2.1).
9. What are the main pillars of general relativity?
   Answer: On the qualitative side, the two main pillars of general relativity are the equivalence principle (see § 1.8.2), which equates gravity to acceleration, and the curvature of spacetime (as an underlying cause for gravitation) due to the presence of mass and energy (see § 1.8.4). On the quantitative and formal side, general relativity rests on the concepts and techniques of differential geometry and tensor calculus to correlate the curvature of spacetime to matter and hence find the metric of spacetime which contains all the information required to describe and quantify gravity such as the geodesic paths which physical objects follow when they are not under the influence of any force (i.e. excluding gravity which is not a force in a technical sense according to general relativity).
10. Explain the need for tensor calculus and differential geometry in the formulation of general relativity.
    Answer: Tensor calculus is needed to formulate the laws of physics (including gravity which is the central subject of general relativity) in compact, elegant and invariant forms (see § 1.8.1 and § 2.2) while differential geometry is needed to determine and describe the geometry of curved spacetime which underlies gravity in general relativity.
11. In general relativity, what is the link between the doctrine of representing gravity by the curvature of spacetime and the redundancy of the concept of force (as applied to gravity)?
    Answer: The link is the general relativistic doctrine that free objects follow geodesic trajectories. The reason is that since gravity is represented by the curvature of spacetime (i.e. gravity is embedded in the structure of spacetime) and because free objects follow geodesics which are determined by this curvature then any effect of gravity will be taken into consideration by the structure of spacetime which defines these geodesics and hence the paradigm of (gravitational) force is redundant.

1.2 Historical Issues and Credits

1. State some limitations of the classical gravitation theory that motivated the development of general relativity.

   Answer: Limitations include:

   • Lack of temporal dependency. A consequence of this is the propagation of gravitational effects (or interaction) with seemingly infinite speed which seems physically non-viable, moreover it contradicts the special relativistic premise that the speed of light c is the ultimate limit of any physical speed.

   • Failure in some predictions such as the perihelion precession of Mercury.

   • Non-invariance of gravitational force across inertial frames under Lorentz transformations and hence it is not a proper physical law according to the relativity principle of special relativity.[1]

2. Discuss the twists and turns in the development of general relativity.

   Answer: The initial attempts in developing this theory were motivated by and focused on generalizing special relativity (which is restricted to inertial frames) to include accelerating frames, but the subsequent events changed the objective and course of action to become the development of a gravitational theory although the initial banner of "generalizing special relativity" remained. In fact, this is a source of inconsistency and confusion particularly with regard to the relation between special relativity and general relativity and the objective of invariance as well as the more fundamental issue of the duality in the qualification of general relativity as a gravity theory on one hand and as a "General Theory" on the other hand.

   To be more clear, we believe that the general relativity project was initially staged as a mission for developing a comprehensive theory of physics (or a "General Theory") from the perspective of spacetime and the appropriate physical transformations across reference frames such that the laws of physics can be formulated in forms that apply in all types of frame (i.e. accelerating as well as inertial). As we know, Lorentz mechanics (or special relativity) unified in practice the branches of mechanics and electromagnetism (and possibly other branches of physics at least through the claim of generality of invariance according to the relativity principle within special relativity and even before; see our book "The Mechanics of Lorentz Transformations")[2] and made them invariant under the Lorentz transformations across inertial frames. This, in essence, means that Lorentz mechanics is a "General Theory" (considering the claimed generality of the invariance according to the relativity principle which does not distinguish mechanics and electromagnetism from other branches) as far as inertial frames are concerned. So, the initial objective of the "general relativity" project was to generalize this "General Theory" and extend it to include accelerating frames. However, this generalization was not seen sufficient to make this theory a "General Theory" because allegedly we still have another type of frames, i.e. "gravitating frames". In fact, this is a major twist in the development of this project and subsequent formulation of the theory of general relativity. It simply reflects the confusion and blur of vision in the early stages which left its marks on the final product of this project, i.e. general relativity. This is because while gravity is a physical phenomenon (like any other phenomenon of mechanics and electromagnetism for instance) that should be subject to the rules of spaces and transformations between frames, we are now assuming that gravity is something different from other physical phenomena by proposing (explicitly or implicitly) that gravity is not a physical phenomenon that takes place in spacetime and is contained in it like any other phenomenon but it is a determining factor of the nature of spacetime and the frames that coordinate it. This seems to be the result of Einstein's desire to combine "the happiest thought of his life" (i.e. the alleged equivalence between acceleration and gravity) with the project of finding a "General Theory" by extending Lorentz mechanics to include accelerating frames (noting that such combination will supposedly address indirectly the problem of non-invariance of gravity under the Lorentz transformations as well). In fact, if the initial project of generalizing Lorentz mechanics to accelerating frames and hence finding a "General Theory" for all types of frame (i.e. inertial and non-inertial), was grown and developed normally and naturally without this confusion then gravity would have been treated like any other physical phenomenon and hence the laws of physics (including gravity) would have been formulated in an invariant way for all types of frame (i.e. inertial and non-inertial) with no need for distorting the spacetime by gravity and making gravity a determining factor of the nature of spacetime and an element in the formation of frames of reference.[3] In other words, while keeping the underlying spacetime flat we extend the invariance to include even gravity and even accelerating frames by reformulating the gravity or/and by developing a more general type of transformations than the Lorentz transformations.

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[1] The non-invariance of classical gravitational force with respect to Lorentz transformations is demonstrated in the literature by technical arguments. However, it should be evident because it is a 3-vector while any Lorentz invariant vector should be a 4-vector (see our book "The Mechanics of Lorentz Transformations"). In brief, a 3-vector cannot be Lorentz invariant (since this invariance applies to 4-vectors in the 4D spacetime of Lorentz mechanics). This may also be established by noting that the absence of a temporal dependency (which leads to action at a distance) is a clear violation of special relativity and hence classical gravitational force cannot be Lorentz invariant.

[2] The failure of Lorentz mechanics to unify gravity under this invariance requirement will be addressed later.

In brief, there is no reason (other than the aforementioned historical reason despite the opposite claims) to single out gravity and treat it differently. If the modern theory of gravity (as represented currently by general relativity) was naturally developed then we can in principle formulate all physical laws (including gravity) in a flat spacetime (like the classical space-time or the Lorentzian spacetime) and gravity will be like any other force that takes place in this flat spacetime instead of distorting this spacetime and introducing many complexities and potential defects. In this case, a healthy modern gravity theory will be based on the fundamental paradigm of classical gravity (and even Lorentz mechanics) with certain extensions, generalizations and improvements some of which may be taken from the essential elements of general relativity as a gravity theory.

3. Discuss the failure of Lorentz mechanics to include gravity under the invariance requirement (to be like mechanics and electromagnetism).

   Answer: In brief, this is one of the issues related to the historical development of general relativity which is based on the distinction between gravity and other physical phenomena. Otherwise, gravity could in principle have also been modified (as classical mechanics was modified earlier) to become Lorentz invariant. In other words, addressing the non-invariance of classical gravity could have been done more naturally not by adopting the new paradigm of curved spacetime of general relativity but by modifying classical gravity within the same paradigm of flat spacetime of Lorentz mechanics as we did in addressing the problem of non-invariance of classical mechanics (in its original form) where we introduced changes on certain concepts, relations and quantities (e.g. momentum) and created new ones (e.g. mass energy) and hence made classical mechanics (in its modified form) Lorentz invariant. Accordingly, if the "general relativity" project was developed naturally then we should have witnessed in the first stage the extension of the Lorentz invariance across inertial frames to include gravity. The second stage of creating a "General Theory" will then be the extension of the invariance of all laws (possibly under a new set of transformations) to include even non-inertial frames while keeping the spacetime flat.

   We should finally remark that the invariance of gravity (as well as any other physical law) under Lorentz transformations should be guaranteed by the presumed validity of these transformations.[4] In other words, if Lorentz transformations are the correct transformations then all the laws of physics (including gravity in its modified correct form) should transform invariantly under them, and hence if we found that Lorentz transformations cannot transform gravity invariantly then this should indicate that these transformations are not the right set of transformations and hence we should look for another set of transformations under which all the laws transform invariantly. The justification of this logic is the rejection of the Galilean transformations because although they transform classical mechanics invariantly they do not transform electromagnetism invariantly and that is why we threw them away and replaced them with Lorentz transformations alleging that these new transformations transform all the laws of physics invariantly (although to avoid the embarrassment with gravity we were forced to claim that gravity is different and it is not a phenomenon in spacetime but it is an attribute of spacetime). In other words, if we have right to reject the Galilean transformations because they can transform only some (not all) of the physical laws invariantly then we should have right to reject Lorentz transformations for the same reason if they proved to be incapable of transforming some laws (particularly gravity) invariantly. By singling out gravity to avoid the problem of invariance we simply created a mess and introduced unnecessary complications and potential faults.

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[3] By this, we are addressing the failure of gravitational force to be invariant under the Lorentz transformations which is used as an excuse and cause for developing general relativity as a theory *of spacetime* instead of re-formulating gravity invariantly or/and replacing the Lorentz transformations with a more general type of transformations (that can transform all the laws including gravity invariantly) and hence developing a theory of gravity *in spacetime*.

[4] It is claimed in the literature that many unsuccessful attempts have been made to make Newton's law of gravity compatible with special relativity. However, this is not a sufficient excuse for treating gravity differently.

Note: the existence of a linearized version of general relativity (see § 3.4) which can be seen as a gravitational field theory in the Minkowski spacetime of special relativity (or Lorentz mechanics) should support (in principle at least) the possibility of constructing a gravity theory on a flat-spacetime background and within the fundamental formalism of Lorentz mechanics. The fact that this linearized version is an approximate theory or its validity is restricted to weak fields does not affect the principal objective of our argument since the construction of an "exact" and more general gravity theory in a flat spacetime remains a possibility when we already have a gravity theory in such spacetime (even if the theory is approximate and restricted). After all, if we dig deep then even general relativity (in its full formalism and not only in its linearized form) is an approximate theory based on many simplifications, restrictions and compromises (and being a theory in curved spacetime does not affect its nature from this perspective). We should also note that there are invariant formulations of the theory of gravity (whether classical or non-classical) in flat spacetime.

4. Outline the main limitations of Lorentz mechanics (or special relativity) with regard to the requirement of the invariance of physical laws and explain and assess the general relativistic methodology in addressing these limitations.

Answer: There are two main limitations:

- The non-invariance of gravity under the Lorentz transformations.
- The non-invariance of physical laws in general across accelerating frames (since the domain of Lorentz mechanics is restricted to inertial frames).

The methodology of general relativity in addressing these limitations is to classify gravity as a spacetime issue rather than a physical issue (and hence it is not subject to the invariance requirement) and to use the equivalence principle (with alleging the applicability of Lorentz mechanics locally) to claim the invariance of physical laws across all frames (i.e. inertial, accelerating and gravitating).

Our view is that this is a distorted methodology. The ideal methodology in addressing these limitations is to extend in the first stage the invariance in Lorentz mechanics to include gravity (either by modifying gravity like mechanics or/and by modifying or replacing Lorentz transformations), and to generalize in the second stage the transformations to include accelerating frames (where this generalization could include modification of some or all laws to comply with the invariance requirement under the proposed general transformations).

5. Briefly outline the role of Einstein in the development of general relativity and in the development of a general theory of space and time (as well as gravitation metric theory).

Answer: Regarding the development of general relativity, Einstein was the main driving force in the search for a new general theory of space and time (which will be known as general relativity with gravity being its central theme), but it is important to note that it was impossible for Einstein to succeed in his attempts without fundamental ideas and techniques and essential help and advice from other scholars such as Grossmann and Hilbert. Regarding the wider picture of developing a general theory of space and time (as well as gravitation metric theory), general relativity was not the only project under development at the time of developing general relativity since there were projects (other than general relativity project) in which Einstein has no involvement. In fact, some of those projects preceded general relativity project and they provided inspiration, guidance and essential scientific ideas and techniques for the development of general relativity.

6. Why we question the priority of Einstein in developing the final form of the Field Equations of general relativity and attribute the main credit to Hilbert?

Answer: We may justify this by the following:

- Einstein lacks sufficient mathematical knowledge and skills to develop these equations (independently at least) while Hilbert is undisputedly a great mathematician. So, the formulation of these equations by Hilbert seems more logical and comprehensible.

• The track record of Einstein reveals his tendency to hide his sources (e.g. ignoring Poincare in his special relativity paper which does not include any formal reference). So, it is not odd for a man with such a character and record to do this again.

• There are historical indications that the final form of the Field Equations has been communicated to Einstein by Hilbert during their exchange of letters about this issue in the final stage of developing this theory. In fact, there seems to be an implicit confession by Einstein in these letters about Hilbert priority in developing the final form of these equations where Einstein is quoted to write in one of his letters to Hilbert "The difficulty was not to find generally covariant equations for the $g^{\mu\nu}$; this is easy with the help of the Riemann tensor. What was difficult instead was to recognize that these equations form a generalization, and that is a simple and natural generalization of Newton's law" because in this quote Einstein seems to implicitly acknowledge the contribution of Hilbert in developing this formal aspect but he claims that what is important is his own contribution in recognizing that these equations form a generalization. So in brief, Einstein is implicitly saying to Hilbert: what is important is not your formal part of the work but it is my idea in recognizing this generalization. There are other implicit indications in these letters to the priority of Hilbert.

• Einstein made a breakthrough only after his contact and collaboration with Hilbert. If Einstein was able to develop these equations independently then we may expect him to be able to do this before because he was dedicating himself to the development of this theory and formulating these equations for long time. The occurrence of this in a rather short period of time following this contact should suggest that the development of these equations is the work of Hilbert or at least Hilbert is the main contributor to the development of the final form of the theory as represented by the Field Equations.

• It is hard to believe that Einstein was willing to share his research and expose his ideas if he was able to work independently on developing these formal aspects of the theory. This applies to his collaboration with Hilbert as well as with Grossmann and other collaborators.[5]

7. Is general relativity the first metric theory of gravity?
   Answer: No. It is neither the first nor the last. In fact, there are many metric theories of gravity before and after general relativity.

1.3 General Terminology

1. Make a brief comparison between the space of special relativity and the space of general relativity.
   Answer: We note the following:
   • Both spaces are 4D spacetime manifolds with 1 temporal dimension and 3 spatial dimensions.
   • The space of special relativity is flat while the space of general relativity is curved.
   • The space of special relativity is Minkowskian while the space of general relativity is Riemannian (or rather pseudo-Riemannian).
   • The space of special relativity is void of matter and energy (as source of gravity) while the space of general relativity is not. In other words, gravity exists in the space of general relativity but not in the space of special relativity.
2. Clarify the issue of line element and quadratic form in general nD space and in the spacetime of general relativity.
   Answer: The infinitesimal line element of a general nD space is usually labeled with ds and we call its square, i.e. $(ds)^2$, the quadratic form.[6] For the 4D spacetime s is replaced by σ and hence $d\sigma$ and $(d\sigma)^2$ are used instead of ds and $(ds)^2$.

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[5] Based on inspection of the history of Einstein, we may classify his research projects into three main categories: projects of his own that he can manage independently like the photoelectric investigation, projects of his own that he cannot manage independently like the general relativity investigation, and projects of others that he joined like the Bose-Einstein investigation. We can claim that he shared (or collaborated with) others in the last two categories out of necessity and not out of choice because he was generally adamant in protecting his work and ascertaining his precedence (unlike some of his contemporaries like Poincare and Hilbert).

[6] To be more accurate, the quadratic form is the mathematical expression (i.e. $g_{ij}dx^i dx^j$) that represents $(ds)^2$.

3. What is the essence of the definition (which is given in the text) of flat space?
Answer: The essence of this definition is that a space is flat *iff* it can be coordinated by a Cartesian coordinate system (where "Cartesian" here includes "quasi-Cartesian" or "pseudo-Cartesian" when some of the diagonal elements of the metric tensor are negative as in the case of Minkowski space and metric).
4. State some of the distinctive properties of pseudo-Riemannian spaces.
Answer: We note the following:
• Some of the diagonal elements of the metric tensor of pseudo-Riemannian space are positive and some are negative (assuming the metric tensor is diagonal).
• The quadratic form $(ds)^2$ is not positive definite and hence its line element ds can be zero or imaginary (as well as positive).
5. Give a detailed and formal definition of "spacetime interval" in special relativity.
Answer: In the flat spacetime of special relativity (i.e. Minkowski space), spacetime interval σ between two events, $V_1(\mathbf{x}_1)$ and $V_2(\mathbf{x}_2)$ connected by a straight path (or world line) in the spacetime is defined as:
$$\sigma \equiv \Delta\sigma = \sqrt{(\Delta x^0)^2 - (\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2}$$
where $\Delta x^\mu = x_2^\mu - x_1^\mu$ and $\mu = 0, 1, 2, 3$. When the path is curved, the spacetime interval is defined as the integral of the line element $d\sigma$ along the path, that is:
$$\sigma = \int_P d\sigma$$
where $d\sigma = \sqrt{(dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2}$ and P symbolizes the path that connects the two events. In fact, this definition of spacetime interval in special relativity is just a special case or an instance of the definition of spacetime interval in general relativity (which is given in the text) noting that the Minkowski spacetime is flat and its metric is given by $\operatorname{diag}[1, -1, -1, -1]$.
6. What "free particle" means in special relativity and in general relativity? What is common to both? Also, link this to the concepts of inertia and gravity and hence find a more fundamental common factor.
Answer: In special relativity free particle means it is free of the influence of any force including gravity (noting that gravity does not exist in the spacetime of special relativity), while in general relativity it means it is free of the influence of any force other than gravity (noting that gravity according to general relativity is not a force in a technical sense).
The common to both is that free particle is not under the influence of any force (noting that gravity is not a force). Accordingly, free particles follow geodesic paths in both spaces (i.e. spaces of special and general relativity) although these geodesics are straight in special relativity and are generally curved in general relativity.
The link is that free particle in special relativity is under the influence of inertia only while free particle in general relativity is under the influence of inertia or/and gravity only and this should be a common factor according to the equivalence principle which essentially abolishes the difference between inertia and gravity.
7. Give examples of massive free objects (i.e. under the influence of inertia or/and gravity alone) in the spacetime of general relativity.
Answer: Example of free object under the influence of inertia alone is a bullet fired or released at infinity where there is no gravity. Example of free object under the influence of gravity alone is a bullet released from rest in the gravitational field. Example of free object under the influence of inertia and gravity alone is a bullet fired in the gravitational field. In fact, the difference between the last two cases may be disputed because in both cases the massive object is actually under the influence of inertia and gravity together and this may be justified by the claim that any massive object should keep its inertia in gravitational fields regardless of its initial kinematical state. The last two cases may also be seen as examples of being under the influence of gravity alone and this may be justified by the claim that being under the influence of inertia and being under the influence of gravity are disjoint (i.e. mutually exclusive) cases. In fact, all these potential differences should lose most of their significance in the light of the equivalence principle which effectively unifies inertia and gravity.

8. Is there a difference between "free" and "freely falling"?
   Answer: Being free may be seen as more general than being freely falling. For example, according to this view an object released from rest and it is falling toward a gravitating body is free and freely falling, while a projectile fired in a gravitational field is free but it is not freely falling in the usual sense. Similarly, an object moving in the Minkowski spacetime or moving very far away from the source of gravity in the spacetime of general relativity with no influence of any force is also free but it is not freely falling in the usual sense. So, "free" may suggest being under the influence of inertia or/and gravity while "freely falling" may suggest being under the influence of gravity specifically. However, we should insist that this is not a convention and hence in the presence of gravity "free" and "freely falling" are generally used interchangeably unless some indications suggest otherwise. Anyway, this is a trivial matter and the context should clarify the situation (if such a clarification is needed at all). We finally note that this question is based on similar considerations to those of the previous question and hence its purpose is to provide more clarity.
9. Compare inertial mass and gravitational mass.
   Answer: We note the following:
   • Both masses are attributes of massive (or material) objects such as atoms and planets (noting that the effective mass of massless objects is not usually described as inertial or gravitational).
   • Both masses represent quantitative measure of the amount of matter contained in massive objects.
   • While inertial mass is related to the state of rest and motion of the object, gravitational mass is related to the pulling force that a massive object experiences in the presence of another massive object. In other words, inertial mass is a kinematical concept while gravitational mass is a dynamical concept.
   • Inertial mass is defined and quantified classically by Newton's second law while gravitational mass is defined and quantified classically by Newton's law of gravitation.[7]
10. What is the relation between inertial mass and acceleration?
    Answer: Inertial mass is a quantitative measure of the resistance to acceleration.
11. Give brief definitions of the following terms: pseudo-Riemannian space, null geodesic, tidal force, free fall gravitation, and test particle.
    Answer:
    Pseudo-Riemannian space is a manifold characterized by having a symmetric rank-2 metric tensor that describes its geometry and hence it is like Riemannian space in this regard. However, its metric[8] (unlike the metric of Riemannian space) is not positive definite.
    Null geodesic is a geodesic path in the 4D spacetime (whether flat of special relativity or curved of general relativity) with a vanishing spacetime interval, i.e. $d\sigma = 0$ and hence $\sigma = 0$. Accordingly, null geodesic is the trajectory (or world line) of light (or massless objects to be more general) in the 4D spacetime (assuming it is free of the influence of non-gravitational forces and interactions such as being reflected or diffracted).
    Tidal force is a differential force caused by the gradient of non-uniform force field.
    Free fall gravitation is the motion in spacetime under the influence of gravity alone.
    Test particle is a point-like gravitated object whose mass is so tiny that it does not affect the gravitational field (created by the gravitating object) which the test particle is under its influence.
12. Give a mathematical condition that distinguishes between Riemannian space and pseudo-Riemannian space.
    Answer: In Riemannian space we must have $(ds)^2 > 0$ while in pseudo-Riemannian space we have $(ds)^2 > 0$ or $(ds)^2 = 0$ or $(ds)^2 < 0$.
13. Give some examples of the significance of the locality condition in various physical contexts.
    Answer: For example:
    • Locality with respect to space/spacetime geometry means that the part under consideration of space/spacetime is approximately flat, i.e. Euclidean/Minkowskian.
    • Locality with respect to force means that the effect of tidal forces is negligible.

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[7] However, this does not mean they are classically different (see § 1.8.2).

[8] Or rather its quadratic form which is based on its metric. In fact, some authors follow a convention that "metric" is for the quadratic form and "metric tensor" is for the tensor.

• Locality with respect to frame means that the frame is approximately inertial.
• Locality with respect to coordinate system means that a confined Cartesian coordinate system can be used.
• Locality with respect to time means that the effects of curvature, non-linearity, variation, etc. are negligible over the given period of time.

14. Give common examples of how curved spaces of various dimensionality look flat locally. Also justify your findings.
Answer: A circle (which is a 1D curved space)[9] looks straight when we consider a tiny piece of it so it is locally flat. Also, a sphere (which is a 2D curved space) looks plane when we consider a tiny piece of it so it is locally flat. Hence, we can conclude that curved spaces of higher dimensionality (e.g. the spacetime of general relativity) are also locally flat in this sense. The justification of this phenomenon of "being flat locally" is that although the curvature of space is a local property (i.e. it is defined locally using local parameters) it is also a global property for the space where the effects of local curvatures accumulate and grow in the eye of the observer and hence the curvature of the space becomes more intense and obvious as the scale grows larger and larger and eventually the curvature becomes most intense and obvious at global level (which gives the impression of flatness at local level due to the relative smallness of curvature at this level). However, we should note that this is not always the case but it is the common case. We should also note that we are considering Riemannian spaces and hence from a more technical perspective the local flatness of space is a property of Riemannian spaces (including pseudo-Riemannian spaces).

15. Give mathematical definitions for "locally Euclidean space" and "locally flat spacetime". Also explain "local Cartesian coordinate system" and "local inertial frame".
Answer:
Locally Euclidean space: in the immediate vicinity of a given point in the space the metric tensor g_{ij} of the space can be expressed (possibly with the help of some coordinate transformation) as $g_{ij} \simeq \delta_{ij} + \varepsilon_{ij}$ where δ_{ij} is the Kronecker delta tensor and $|\varepsilon_{ij}| \ll 1$.
Locally flat spacetime: in the immediate vicinity of a given point (or event) in the spacetime the metric tensor $g_{\mu\nu}$ of the spacetime can be expressed (possibly with the help of some coordinate transformation) as $g_{\mu\nu} \simeq \eta_{\mu\nu} + \varepsilon_{\mu\nu}$ where $\eta_{\mu\nu}$ is the Minkowski metric tensor and $|\varepsilon_{\mu\nu}| \ll 1$.
Local Cartesian coordinate system is a Cartesian coordinate system employed in the immediate vicinity of a given point in a space that is locally Euclidean at that point.
Local inertial frame is an inertial frame employed in the immediate vicinity of a given point in a spacetime that is locally flat at that point.

16. Use the answer of exercise 15 to give mathematical conditions for local flatness in Riemannian and pseudo-Riemannian spaces.
Answer:
Local flatness in Riemannian space: $g_{ij} \simeq \delta_{ij} + \varepsilon_{ij}$ where g_{ij} is the metric tensor of the locally flat region of space Ω, δ_{ij} is the unity tensor (or Kronecker delta tensor) of a Cartesian system coordinating Ω, and ε_{ij} is a perturbation tensor (with $|\varepsilon_{ij}| \ll 1$).
Local flatness in pseudo-Riemannian space: $g_{ij} \simeq \Delta_{ij} + \varepsilon_{ij}$ where g_{ij} is the metric tensor of the locally flat region of space Ω, Δ_{ij} is the metric tensor of a pseudo-Cartesian system coordinating Ω, and ε_{ij} is a perturbation tensor (with $|\varepsilon_{ij}| \ll 1$).
Note: the above forms of the metric tensor g_{ij} may require coordinate transformation.

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[9] In fact, being curved is from an extrinsic perspective since 1D spaces have no intrinsic curvature. Accordingly, the argument in this answer is loose in this regard and hence it is for pedagogical purpose. The intrinsic flatness of 1D spaces can be easily established from the fact that a presumed Riemann-Christoffel curvature tensor of such spaces can only have one component R_{1111} (since $n = 1$) and by the anti-symmetry properties of this tensor it should be zero (see § 2.3 and § 2.10).

1.4 General Conventions, Notations and Remarks

1.5 Classical Gravity

1. Find the gravitational force field on the surface of the Earth (assuming it is a perfect sphere with spherically symmetric mass distribution).
Answer: We use the equation $\mathbf{g} = -\frac{GM}{r^2}\mathbf{e}_r$ where M is the Earth's mass ($\simeq 5.9722 \times 10^{24}$ kg) and r is its mean radius ($\simeq 6371000$ m), that is:
$$\mathbf{g} = -\frac{GM}{r^2}\mathbf{e}_r \simeq -\frac{6.674 \times 10^{-11} \times 5.9722 \times 10^{24}}{(6371000)^2}\mathbf{e}_r \simeq -9.8\mathbf{e}_r \,\text{N/kg}$$
As we see, the magnitude of the field (i.e. 9.8) is the familiar gravitational "acceleration" of the Earth. The "acceleration" tag originates from equating Newton's gravity law to Newton's second law (i.e. $m\mathbf{a} = m\mathbf{g}$) and hence the gravitational force field $\mathbf{g}$ in Newton's gravity law corresponds to the acceleration $\mathbf{a}$ in Newton's second law.
2. Obtain a mathematical expression for the gravitational potential Φ.[10]
Answer: In normalized spherical coordinates the gradient of a scalar function Φ is given by (see B3):
$$\nabla\Phi = \mathbf{e}_r\partial_r\Phi + \mathbf{e}_\theta\frac{1}{r}\partial_\theta\Phi + \mathbf{e}_\phi\frac{1}{r\sin\theta}\partial_\phi\Phi$$
Now, from Eqs. 12 and 5 (in the text) we have:
$$\mathbf{g} = -\nabla\Phi = -\frac{GM}{r^2}\mathbf{e}_r$$
On comparing the last two equations we obtain:
$$\partial_r\Phi = \frac{GM}{r^2}$$
On integrating the two sides of the last equation noting that the potential, like the gravitational force field, is radial (i.e. it depends only on r) we get:
$$\Phi = -\frac{GM}{r}$$
where the constant of integration is set to zero inline with the convention that the potential is zero at infinity.
We can check our result by using the definition of $\mathbf{g}$ that is:
$$\begin{aligned}\mathbf{g} &= -\nabla\Phi \\ &= -\mathbf{e}_r\partial_r\left(-\frac{GM}{r}\right) - \mathbf{e}_\theta\frac{1}{r}\partial_\theta\left(-\frac{GM}{r}\right) - \mathbf{e}_\phi\frac{1}{r\sin\theta}\partial_\phi\left(-\frac{GM}{r}\right) \\ &= -\mathbf{e}_r\frac{GM}{r^2} - \mathbf{e}_\theta 0 - \mathbf{e}_\phi 0 \\ &= -\frac{GM}{r^2}\mathbf{e}_r\end{aligned}$$
which is the same as Eq. 5 (in the text). We note that in line 2 we use the expression of gradient in normalized spherical coordinates.
3. Give two properties of conservative vector field.
Answer: It is the gradient of a scalar field. It is irrotational, i.e. its curl vanishes identically.

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[10] In questions like this we assume a single gravitating object with spherically symmetric (or point like) mass distribution.

4. Prove that the gravitational field is conservative.
   Answer: We use the second property in the answer of exercise 3 (i.e. irrotational), that is:

$$\begin{aligned}
\nabla \times \mathbf{g} &= \frac{1}{r^2 \sin\theta} \begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\mathbf{e}_\phi \\ \partial_r & \partial_\theta & \partial_\phi \\ g_r & rg_\theta & r\sin\theta g_\phi \end{vmatrix} \\
&= \frac{1}{r^2 \sin\theta} \begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\mathbf{e}_\phi \\ \partial_r & \partial_\theta & \partial_\phi \\ g_r & 0 & 0 \end{vmatrix} \\
&= \frac{1}{r^2 \sin\theta} \left(0\mathbf{e}_r + r\mathbf{e}_\theta \partial_\phi g_r - r\sin\theta\mathbf{e}_\phi \partial_\theta g_r\right) \\
&= \mathbf{0}
\end{aligned}$$

   where in line 1 we use the definition of curl in spherical coordinates (see B3), in line 2 we use the fact that the gravitational field is radial and hence it has no θ or ϕ components (i.e. $g_\theta = g_\phi = 0$) according to Eq. 5 (in the text), and line 4 is because g_r (which is equal to $-\frac{GM}{r^2}$ according to Eq. 5) is independent of ϕ and θ and hence $\partial_\phi g_r = \partial_\theta g_r = 0$. Accordingly, the gravitational field is irrotational and therefore it is conservative.
   Note: in fact, the answer of exercise 2 can also be seen as another proof (based on the first property in the answer of exercise 3).
5. Verify the divergence theorem for the gravitational field of a gravitating body assuming it is a perfect sphere with spherically symmetric mass distribution.
   Answer: According to the divergence theorem we have:

$$\int_S \mathbf{g} \cdot \mathbf{n}\, dA = \int_\Omega \nabla \cdot \mathbf{g}\, dV \tag{1}$$

   Now, we found in the text that the left hand side of Eq. 1 is equal to $-4\pi GM$, i.e.

$$\int_S \mathbf{g} \cdot \mathbf{n}\, dA = -4\pi GM$$

   Regarding the right hand side of Eq. 1, there is a technical problem that is the divergence of an inverse square field (excluding the origin) is zero. However, because of the spherical symmetry of the mass distribution we can consider the gravitating body as a point mass at the origin and hence the right hand side of Eq. 1 can be expressed in terms of the Dirac delta function $\delta^3(\mathbf{r})$ as:

$$\begin{aligned}
\int_\Omega \nabla \cdot \mathbf{g}\, dV &= \int_\Omega \nabla \cdot (-\nabla\Phi)\, dV \\
&= \int_\Omega \nabla \cdot \left(-\nabla\left[-\frac{GM}{r}\right]\right) dV \\
&= GM \int_\Omega \nabla^2 \left(\frac{1}{r}\right) dV \\
&= GM \int_\Omega -4\pi\delta^3(\mathbf{r})\, dV \\
&= -4\pi GM \int_\Omega \delta^3(\mathbf{r})\, dV \\
&= -4\pi GM
\end{aligned}$$

   where in line 1 we use $\mathbf{g} = -\nabla\Phi$, in line 2 we use $\Phi = -\frac{GM}{r}$ which we obtained in exercise 2, in line 3 we use the definition of the Laplacian operator as the divergence of gradient, and in lines 4 and 6 we use some of the properties of the Dirac delta function (the interested reader is advised to consult the literature on this issue). Accordingly, the left and right hand sides of the divergence theorem (i.e. Eq. 1) are identical and hence the divergence theorem is verified in this case.

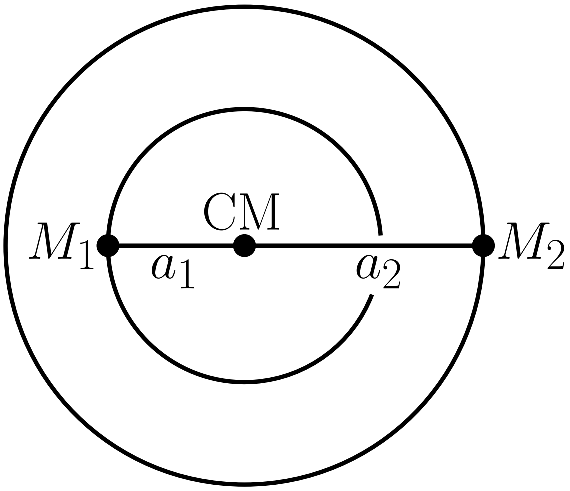

Figure 1: A simple sketch representing a binary system whose objects of mass M_1 and M_2 revolve around their center of mass CM following circular orbits of radius a_1 and a_2 respectively.

1.5.1 Planetary Motion

1. What is the relation between Kepler's laws and Newton's law of gravity?
   Answer: We note the following:
   • Kepler's laws are consequences of Newton's law of gravity (supported by Newton's second law of motion) and hence they can be derived from Newton's law (as we did in the Problems). However, historically Kepler's laws provided the insight for Newton to derive his law of gravitation and hence Newton's law is a product of Kepler's laws.
   • Kepler's laws are about orbital motion[11] and hence they are restricted in this regard while Newton's law of gravity is general since it applies to all sorts of motion and interaction by the force of gravity.
   • Kepler's laws are about motion while Newton's law of gravity is about force. In other words, Kepler's laws are essentially kinematical (i.e. description of motion) while Newton's law is dynamical (i.e. causes of motion).
2. Show that for a binary system whose objects revolve around their center of mass Kepler's third law keeps its form with a standing for the semi-major axis of the ellipse that represents the orbit of one object in the rest frame of the other object while M stands for the sum of the masses of the two objects. (Hint: you can simplify the analysis by assuming circular motion.)
   Answer: Let have two objects of mass M_1 and M_2 and they revolve around their center of mass where a_1 and a_2 are their distances from the center of mass. To simplify the analysis we assume that both objects have circular orbit (see Figure 1). We should also note that throughout their motion the straight line connecting the two objects passes through their center of mass, and hence the distance a that separates them will be given by: $a = a_1 + a_2$. Also, from the definition of the center of mass we have: $M_1 a_1 = M_2 a_2$. Accordingly:

$$\begin{aligned} M_2 a_2 &= M_1 a_1 \\ a_2 &= \frac{M_1 a_1}{M_2} \\ a_2 &= \frac{M_1 (a - a_2)}{M_2} \end{aligned}$$

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[11] Historically, Kepler's laws are about orbital motion of planets in the solar system although they have general validity for orbital motion within the classical conditions. This general validity is based on the general validity of Newton's law of gravity which underlies these laws.

$$
\begin{aligned}
a_2 &= \frac{M_1 a}{M_2} - \frac{M_1 a_2}{M_2} \\
a_2 + \frac{M_1 a_2}{M_2} &= \frac{M_1 a}{M_2} \\
a_2 \left(1 + \frac{M_1}{M_2}\right) &= \frac{M_1 a}{M_2} \\
a_2 \left(\frac{M_1 + M_2}{M_2}\right) &= \frac{M_1 a}{M_2} \\
a_2 &= \frac{M_1 a}{M_1 + M_2} \qquad (2)
\end{aligned}
$$

Now, according to Newton's gravity law the magnitude of the force exerted on M_2 is $\frac{GM_1M_2}{a^2}$. Moreover, M_2 is executing a circular motion of radius a_2 around the center of mass and hence it must be subject to a centripetal force of magnitude given by $M_2 a_2 \omega^2$ where ω is the angular speed. By equating these two forces we obtain:

$$
\begin{aligned}
M_2 a_2 \omega^2 &= \frac{GM_1M_2}{a^2} \\
M_2 \frac{M_1 a}{M_1 + M_2} \omega^2 &= \frac{GM_1M_2}{a^2} \\
\omega^2 &= \frac{G(M_1 + M_2)}{a^3} \\
\frac{4\pi^2}{T^2} &= \frac{G(M_1 + M_2)}{a^3} \\
T^2 &= \frac{4\pi^2 a^3}{G(M_1 + M_2)} \\
T^2 &= \frac{4\pi^2 (a_1 + a_2)^3}{G(M_1 + M_2)}
\end{aligned}
$$

where in line 2 we use Eq. 2 while in line 4 we use the definition of angular speed (with T being the periodic time). The last equation shows that Kepler's third law keeps its form with a standing for the semi-major axis of the ellipse[12] (which is a circle in this case) that represents the orbit of one object in the rest frame of the other object while M stands for the sum of the masses of the two objects.
Note 1: the objects of any binary orbiting system revolve around their center of mass and hence the last equation is general while the equation $T^2 = \frac{4\pi^2}{GM}a^3$ (with M being generally assumed to represent the mass of the more massive object) is an approximation which is justified when the mass of the more massive object (i.e. the gravitating object) is much bigger than the mass of the other object (i.e. the gravitated or test object).
Note 2: the derivation in this exercise is another way for showing that Kepler's third law is a consequence of Newton's laws; a result that we obtained in one of the Problems.

1.6 General Relativity versus Classical Gravity

1. What are the manifolds that underlie the classical theory of gravity and general relativity?
   Answer: The manifold of classical gravity is a flat 3D Euclidean "spatial" space while the manifold of general relativity is a curved 4D Riemannian spacetime.
2. Outline the different philosophical views which the classical theory of gravity and general relativity rest upon.
   Answer: In the classical theory gravity is an influence exerted by matter on matter (and possibly

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[12] This represents a generalization of the result that we obtained from our simplified analysis which is based on assuming circular motions. The justification of this generalization should be sought in the literature of classical mechanics.

on radiation as well) through the intervening gravitational field, while in general relativity gravity is an influence exerted by matter and energy on the geometry of the surrounding spacetime and this geometry affects the behavior of massive and massless physical objects in the spacetime.

1.7 General Relativity versus Special Relativity

1. Give examples of common features between special relativity and general relativity.
Answer: For example:
• Both theories use 4D spacetime to describe physical phenomena and formulate physical laws and hence both are theories of spacetime. However, while the spacetime in special relativity is flat, the spacetime in general relativity is curved due to the presence of matter and energy.
• Both theories rest on the principle of invariance of physical laws although this principle is more general in general relativity.
• In both theories the speed of light is a constant, invariant and ultimate speed although this applies globally in special relativity and locally in general relativity.
• In both theories the results and rules of Lorentz mechanics apply although in general relativity they apply only locally.
• In both theories we have timelike, lightlike and spacelike intervals.
• In both theories free particles follow geodesic world lines although there is a difference in the meaning of "free" and in the nature of geodesic (see the upcoming note 1).
• Similar quantitative and descriptive techniques are used in both theories, e.g. tensor calculus, 4-tensors, spacetime diagrams, light cones, etc. (see the upcoming note 2).
Note 1: "free" in special relativity means free of any type of force, while "free" in general relativity means free of any type of force except gravity (which is not technically a force). Also, geodesics in special relativity are straight lines due to the flatness of spacetime, while in general relativity geodesics are generally twisted curves due to the curvature of spacetime.
Note 2: although the aforementioned techniques are essentially the same in both theories, there are some differences in their application. For example, tensor calculus is a necessity in general relativity but it is not a necessity in special relativity. Also, in special relativity 4-tensors (and their alike of 4-*somethings*) are with regard to Lorentz transformations, while in general relativity 4-tensors (and their alike) are generally with regard to a more general type of transformations (which may include Lorentz transformations in some cases and in local applications).
2. Give examples of features that distinguish special relativity and general relativity from each other.
Answer: For example:
• The spacetime of special relativity is flat Minkowskian, while the spacetime of general relativity is curved Riemannian.
• Physical quantities and laws are global in special relativity and local in general relativity.
• The frames in special relativity are global (covering the entire spacetime), while frames in general relativity are local (covering patches of spacetime).
• In special relativity Lorentz transformations are used, while in general relativity general coordinate transformations are used.
• The domain of special relativity is inertial frames, while the domain of general relativity is frames in general.
• The spacetime of special relativity excludes gravity, while the spacetime of general relativity includes gravity.
• Special relativity is essentially a physical theory about space and time and their transformations,[13] while general relativity is essentially a geometrical theory of gravity (although it includes generalizations that make it "General Theory").

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[13] The geometric nature of special relativity which is essentially based on the 4D spacetime approach and formalism (as indicated in § 1.2) should not contradict this. In fact, special relativity (or Lorentz mechanics) is born as a physical theory but it was geometrized later by the Minkowski 4D spacetime approach. This is unlike general relativity which is a geometric theory from the beginning and by nature.

• Special relativity is mainly a generalization of classical mechanics (where the Galilean transformations are replaced by the Lorentz transformations), while general relativity as a gravity theory is mainly a generalization of the Newtonian gravity and as a "General Theory" is mainly a generalization of special relativity (see the upcoming note).
• The evidence for special relativity (or rather Lorentz mechanics) is overwhelming, while the evidence for general relativity as a gravity theory is tentative and as a "General Theory" is feeble at the best (if not the evidence is actually against the theory).
Note: it should be obvious that general relativity (as a gravity theory) can be regarded as a generalization of Newtonian gravity by extending the domain of gravity from classical gravitational systems to all types of gravitational systems. This also applies to the formalism where the Field Equation of general relativity can be considered a generalization of Poisson's equation of classical gravity with the help of special relativity and some general principles (see exercise 14 of § 3.2). Similarly, general relativity (as a "General Theory") can be regarded as a generalization of special relativity by lifting the restriction of inertiality to include all types of frame (although with potential restriction by locality) and by including gravity.[14]

3. Compare the fundamental principles which special relativity and general relativity rest upon.
Answer:
Regarding special relativity, it is the validity of Lorentz transformations across inertial frames of a flat Minkowskian spacetime.
Regarding general relativity, it is the validity of the equivalence between gravity and inertia (employing the paradigm of curved Riemannian spacetime whose geometry is determined by matter and energy) and the validity of special relativity locally.
4. List some of the features (such as foundations, implications and predictions) of special relativity and the features of general relativity.
Answer:
Regarding special relativity, we have for example: merging of space and time into spacetime, invariance of physical laws across inertial frames, global nature of frames, constancy and invariance of speed of light globally, flatness of spacetime, validity of Lorentz transformations, free motion along (straight) geodesics, kinematical time dilation, kinematical (i.e. Doppler) frequency shift, kinematical length contraction, relativity of simultaneity, relativity of co-positionality, and mass-energy equivalence.
Regarding general relativity, we have for example: merging of space and time into spacetime, invariance of physical laws across all types of frame, local nature of frames, constancy and invariance of speed of light locally, equivalence between gravity and inertia, local validity of special relativity, curvature of spacetime (caused by matter and energy), representation of gravity by curvature of spacetime, sourcing gravity by energy as well as mass, free motion along (curved) geodesics, gravitational time dilation, gravitational frequency shift, gravitational length contraction, and light bending by gravity.
5. What "special" and "general" mean with regard to the two relativity theories? Justify these labels.
Answer: In simple terms, special relativity is special because it deals with spacetime in the absence of gravity while general relativity deals with spacetime in general. This may also be presented as having flat spacetime in special relativity and curved spacetime in general relativity (where flat is seen as a special case of curved). The "special" and "general" tags may also be justified by the fact that special relativity deals with inertial frames specifically while general relativity deals with reference frames in general.
6. It is commonly claimed that special relativity is less general than general relativity. Can we claim that in a sense special relativity is more general than general relativity?
Answer: Yes, we may claim that special relativity (or what we call Lorentz mechanics) in a sense (or from another perspective) is more general than general relativity because special relativity is about transformation of spacetime coordinates which in principle should apply to any physical subject and theory while general relativity is restricted to gravitation.[15] This may also be supported by the fact

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[14] This, of course, should not affect the fact that general relativity is not a generalization of special relativity from other aspects and perspectives (as explained earlier).

[15] In this answer we are considering general relativity from the perspective of gravity theory rather than "General Theory".

that special relativity is supposed to apply locally in general relativity (i.e. special relativity applies in the spacetime of general relativity but general relativity does not apply in the spacetime of special relativity).

7. Assess the relation between special relativity and general relativity and if special relativity is contained in general relativity or they are two different theories.
   Answer: Referring to the arguments that we presented in the text (as well as some exercises) in support of each opinion, we can conclude that both opinions have some validity (as well as invalidity) and this strongly depends on the aspects and perspectives on which these opinions and judgments are based. Hence, when we talk about this issue and its alike we should be more specific to avoid making ambiguous, confusing, controversial and potentially wrong generalizations.

1.8 General Principles

1.8.1 The Principle of Invariance

1. Discuss the "covariance" label that is attached in general relativity to the principle of "invariance".
   Answer: Labeling the principle of invariance of physical laws as the principle of covariance (which means form invariance under certain coordinate transformations) is very common in the literature of general relativity and it has a historical origin. In fact, this is like a name to this principle in general relativity and part of the terminology of this theory. However, to avoid confusion with "covariance" in tensor terminology and to be more general we use "principle of invariance" to refer to this principle in its general and broad sense (which includes its use in general relativity) although we may occasionally use the "principle of covariance" or the "principle of general covariance" to refer to this principle within the framework of general relativity.
   We note that the reader may find certain conventions about the use of these words (e.g. covariant for form invariant and invariant for value or quantity invariant). However, these are not guaranteed to be general conventions and hence they should be limited to their users. So, the reader should be vigilant about the convention of each author. We should also note that some authors (see for example Weinberg in the References) use "general covariance" in a very specific technical meaning (this issue will be discussed in § 1.8.2 where the principle is labeled as the *Principle of General Covariance*).
2. Our assertion that the principle of invariance is an epistemological rather than a scientific principle may be challenged by the fact that certain laws at least (e.g. Newton's second law) are actually invariant regardless of any convention. How do we reply?
   Answer: Our reply is that the observed invariance of Newton's second law, for example, in inertial frames is because we adopted this principle, implicitly or explicitly, in advance before we created our laws (or rather selected what we consider to be laws) and hence this is a result rather than a cause for the principle of invariance. Yes, we should admit that common sense and the spirit of the principles of reality and truth (see B4) dictate the adoption of the invariance principle for the purpose of best adaptation and that is why we see natural philosophers and scholars of early times (i.e. even before formulating this principle explicitly and adopting it consciously by modern scientists) embrace this principle but not necessarily with declaration or even awareness. In fact, we may consolidate our assertion by the fact that this principle is about the physical laws, since its subject is the physical laws, and hence it cannot be a physical law itself. Our view may also be endorsed by the fact that laws like Newton's second law are restricted in validity (or form invariance) to certain types of frame (i.e. inertial) and we still label them as laws, and this should indicate the conventional nature of this principle.
3. How is the principle of invariance implemented in general relativity?
   Answer: It is "implemented" rather arbitrarily and trivially by claiming that all the laws of physics should take the same form in all frames. So, there is no actual scientific substance in this claim or declaration since it is just an announcement that no rule of Nature will be elevated to the rank of physical law unless it is invariant in this sense. In more technical terms, any rule that cannot be expressed as a mathematical tensorial relation is not a law. As we see, this is no more than a claim or

a convention with no real physical substance or actual implementation or demonstration or verification.

4. Question, briefly, the applicability and generality of the principle of invariance in general relativity and if it is really implemented in general under any sort of coordinate transformations.

   Answer: First, we question from a formal perspective if the principle of invariance is really at the heart of general relativity (i.e. it is actually implemented and embedded in this theory) because no actual coordinate transformation is conducted in this theory, which in essence is about gravity, unlike Lorentz mechanics which is all about coordinate transformations. Yes, it is claimed in this theory that the laws of physics equally apply in all frames of reference but this does not mean we are actually conducting coordinate transformations between frames using specific coordinate transformations like Lorentz transformations. Although the requirement of tensor formulation should guarantee the invariance, according to the rules of tensor calculus, this is not the same as implementing the principle within the theory itself as we see for example in Lorentz mechanics. Moreover, there is no conclusive evidence in support of the validity of this principle with regard to all laws of physics. In other words, casting the laws in tensor form may guarantee their invariance mathematically but it does not guarantee their invariance physically because this requires real physical evidence which cannot be provided by a mere mathematical requirement (as noted earlier, the invariance is a necessary but not sufficient condition). So, neither formally nor substantially the principle of invariance is actually implemented or demonstrated or verified technically and physically in general relativity. In brief, it is just a claim or a convention with no actual scientific substance or implementation or demonstration or verification.

5. What is the difference between the invariance principle of special relativity (i.e. the first postulate) and the invariance principle of general relativity?

   Answer: The invariance principle of general relativity is supposed to be an extension to the first postulate of special relativity (i.e. the invariance of physical laws in inertial frames under Lorentz transformations) because this invariance is extended to include gravitational frames and their equivalent accelerating frames. So, the limited special relativistic invariance across inertial frames under Lorentz transformations is replaced in general relativity by the more general invariance across all types of frame under more general coordinate transformations.

   To be more clear, we can say (rather loosely) that the invariance in general relativity is an extension to the invariance in special relativity from three aspects:

   - Domain since the principle is extended from inertial frames to more general frames.
   - Type of transformations since the principle is "extended" from Lorentz transformations to "more general transformations".
   - Subject since the principle is extended from mechanics and electromagnetism[16] to all laws of physics. In fact, this should be particularly true with respect to gravity (though indirectly) because although gravity in general relativity took a special state since it is reclassified from being a physical phenomenon to being a geometric aspect of spacetime, general relativity effectively extended its subject matter to include gravity (which in reality is the main subject of general relativity).

   Note: the answer of this question is about what general relativity is supposed to be rather than what general relativity actually is. It also contains questionable and dubious issues.

6. Briefly discuss the historical development of the principle of invariance with regard to the subject of this principle.

   Answer: The principle of invariance started with mechanics in classical physics where this principle is expressed by the Galilean relativity. The principle was then extended to include electromagnetism as well by the development of Lorentz mechanics. General relativity finally extended this principle to all laws of physics.

   Note: we repeat here what we indicated earlier that the general form of the invariance principle was proposed prior to general relativity (i.e. by the pioneers of Lorentz mechanics and possibly even before) although there was no actual implementation or demonstration or verification of this principle outside the laws of mechanics and electromagnetism (noting also that gravity was excluded anyway). In fact, even the supposed extension in general relativity lacks actual implementation, demonstration and verification in many aspects and details.

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[16] We note that although the invariance of physical laws under the Lorentz transformations according to special relativity (even before Einstein work) includes all physical laws, there was no actual implementation or demonstration of the generality of this invariance outside mechanics and electromagnetism (e.g. gravity). So, this generality was just a claim like the claim in general relativity. Yes, there is later work in many branches of physics on generalizing this invariance, so from a historical perspective the above claim of extension should stand as it is.

7. Lorentz tensor in the frames of 4D spacetime of Lorentz mechanics is the equivalent of the Jacobian matrix in spatial coordinate systems (see B4). So, what is the tensor that corresponds to the Lorentz tensor (i.e. Jacobian matrix) in general relativity?
   Answer: There is no such a specific tensor in general relativity and this should put a question mark on the sensibility and significance of the claimed general invariance in general relativity (which cannot be lifted or dealt with properly by just referring to "general transformations"). In fact, even the existence of such general transformations in all cases and circumstances is not obvious (and hence it requires demonstration and verification).
8. Argue against the invariance principle of physical laws and demonstrate its limited value due to its epistemological and conventional nature.
   Answer: First, we should distinguish between the invariance of the essence of the physical laws and the invariance (or covariance) of the form of the physical laws. While the invariance of the essence should be guaranteed by the lack of physical significance of coordinate systems (as long as the difference between the systems is just conventional) and hence the essence of physical laws should be invariant across conventionally-different coordinate systems (as required by the principles of reality and truth), there is no physical or logical requirement that dictates the invariance of the form of the laws although this is a useful epistemological convention.[17] In brief, as long as we can adapt to the variation of the forms of the physical laws in different coordinate systems (or reference frames) then there is nothing wrong in assuming that the laws can vary in form across different coordinate systems. For example, if we know what the forms of the laws should be in inertial frames and what the forms of the laws should be in non-inertial frames then there should be no problem in having one set of laws in inertial frames and another set of laws in non-inertial frames.[18] In fact, the entire physics that applies restrictively in inertial frames (such as Lorentz mechanics) is based on the acceptance of this form of non-invariance since the laws are already assumed to be different in form in non-inertial frames (e.g. Newton's second law applies only in inertial frames). In fact, we do not even need to have certain classification to the reference frames (e.g. inertial and non-inertial) and hence if a certain "law" takes a certain form in one inertial frame and it takes a different form in another inertial frame (which is distinguished for instance by its speed relative to the first frame) then there is no problem in accepting this "law" because we can apply it (and even transform it) usefully and unambiguously in different frames because we can describe and predict the physical phenomena that take place in these frames by this "law". This can be seen in Lorentz mechanics where the Lorentzian 3-quantities (e.g. 3-force) are not invariant across reference frames since they are frame dependent although their 4D tensorial forms are form invariant. So, despite this non-invariance we still use these 3-quantities as long as we are aware of their restrictions to their particular frames (the reader is also advised to refer to our distinction between physical invariance and mathematical invariance which we discussed in § 6.1 of B4). In fact, the use of non-invariant laws is commonplace in science and engineering such as fluid mechanics.[19] We can therefore conclude that the requirement of the invariance of physical laws should have limited value due to its epistemological and conventional (rather than physical) nature.

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[17] For example, if a physical law does not depend on the orientation in space then there is nothing wrong in expressing this law in two different forms in two coordinate systems (where one of these systems is obtained by static rotation relative to the other system) as long as the two forms provide correct description and predictions in the two systems.

[18] We should note that the difference between inertial frames and non-inertial frames is not conventional if we believe in the existence of absolute frame and its role in identifying and distinguishing these frames. So, the purpose of this example is to highlight the physical insignificance of from invariance in general.

[19] The recent development of tensor calculus (which is largely developed in the end of the 19^{th} century and continued in the 20^{th} century) is an indication of the non-essential nature (or even redundancy) of casting the laws of physics in tensor forms since the entire science prior to this development (including Lorentz mechanics) was based on non-tensorial approach and formulation. This should indicate the difference between the invariance of the content of the laws (which should be essential) and the demand for casting the laws of physics in tensor forms (which is non-essential).

Note 1: regarding our proposal of the necessity of distinguishing between the invariance of the essence of the physical laws and the invariance of the form of the physical laws, some authors seem to suggest a similar proposal where they distinguish between the principle of relativity (which seems to indicate the invariance of essence) and the principle of covariance (which should refer to the invariance of form). For example, we read in Bergmann (see References): "The covariance of equations is the mathematical property which corresponds to the existence of a relativity principle for the physical laws expressed by those equations".

Note 2: it is noteworthy that many of the issues and challenges that we discussed in this subsection (and in this question in particular) are meant to address certain interpretations and claims that are commonly found in the literature. A more technical and rigorous approach will be given later after investigating the principle of equivalence which is closely related to the invariance of physical laws and the principle of invariance (as well as the specific technical meaning of the *Principle of General Covariance*).

9. Discuss the relation between the principle of relativity and the principle of invariance.
   Answer: In our view, the principle of relativity expresses the invariance of the essence of the physical laws across reference frames when the difference between these frames is conventional (or at least this is what the principle of relativity should be).[20] Anyway, the relation between the two principles should depend on the meaning and interpretation of these principles (and the principle of invariance in particular which is characterized by various aspects of vagueness).
10. According to the principle of invariance there should be no privileged frame of reference in which the laws of physics take a particular form. Discuss this briefly.
   Answer: In our view, this should be restricted by the condition that the difference between frames is purely conventional (such as the difference in orientation assuming isotropy of space) and not physical. For example, if we believe in the existence of absolute spacetime (and hence absolute frame) then the difference between inertial and accelerating frames will not be conventional but it is real and physical, and therefore the principle of invariance in its basic formulation should not apply although it may be generalized and extended to include such cases. In fact, this condition is not the only restriction that should be imposed on the invariance principle (but we will not go through these details). We believe that many aspects of the principle of invariance (as well as other principles and aspects of general relativity and even special relativity) are not inspected and investigated properly and thoroughly within the relativity theories and their literature.

1.8.2 The Principle of Equivalence

1. Discuss the following quote which is attributed to Einstein: "The equality of these two masses [i.e. inertial mass and gravitational mass], so differently defined, is a fact which is confirmed by experiments of very high accuracy (experiments of Eotvos), and classical mechanics offers no explanation for this equality. It is, however, clear that science is fully justified in assigning such a numerical equality only after this numerical equality is reduced to an equality of the real nature of the two concepts".
   Answer: We should disagree with Einstein about his claim that classical mechanics offers no explanation for this equality. In fact, Einstein is using his theoretical framework and methodology (which are different from those of classical mechanics) to analyze and assess classical mechanics. In the framework of classical mechanics mass is mass (i.e. a measure of the quantity of matter) whether it is in Newton's second law or in Newton's gravity law or in any other law and this what justifies the use of the same term and symbol, i.e. the concept of mass as a measure of the quantity of matter should be the same from classical viewpoint otherwise different terms and symbols should have been used in these laws. So, it is not a coincident that classical mechanics labels all these with the same word "mass" and uses the same symbol (e.g. m) to refer to mass because this is based on the intuitive classical perception (which originates from our direct experiences with Nature) of mass as a measure of the quantity of matter. Also, it is not a fortunate coincident (as depicted in the literature of general relativity) to find

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[20] If "the principle of relativity" is meant to be that of special relativity specifically (i.e. first postulate) then the frames should be inertial.

inertial mass equivalent to gravitational mass in classical mechanics because this is based on the aforementioned intuitive insight which classical mechanics is based upon. So, what justifies the classical investigations and experiments (refer to the history of the equivalence principle in the Problems) about the equivalence between inertial mass and gravitational mass is the tendency and desire of classical physicists to test and verify this intuitive insight by more rigorous and precise scientific methods. It should also be noted that some of those experiments are directed toward subsidiary issues (rather than the main issue of equivalence) such as possible dependency on the type of material or location in space or time.

In brief, when we start from a general relativistic framework (where initially two types of mass have already been assumed with one being supposedly a property of gravity and the other being a property of acceleration) we may see classical mechanics offering no explanation for this equality. But when we start from a classical framework (where mass is mass and where there is no explicit or implicit presumption of two different types of mass) then the equivalence is self-explanatory although we may still need to verify that the presumption of such a unity or identicality (rather than equivalence) of the "two types" of mass as embedded in our different laws that involve mass is consistent with experiment and hence our intuitive perception of mass as a unique measure of the amount of matter is justified. In fact, from this classical perspective and background these tests and verifications can be seen as indirect tests to the validity of Newton's second law and Newton's gravity law since these laws are based on certain assumptions one of which is the particular dependency on mass where mass is perceived as a unique measure of the quantity of matter. These experiments may also be seen as tests if mass (in one of these laws or the other) has a scaling factor (e.g. 1.00001) whether constant or variable.

Accordingly, the equivalence principle is based on assuming a difference between inertial mass and gravitational mass and this can be denied in the first place according to classical physics, i.e. mass is mass in classical physics and hence such equivalence (or identicality to be accurate) is already embedded in the classical definitions and laws in which mass is involved. Therefore, the equivalence principle at the best is irrelevant if it is not wrong (since equivalence is based on an implicit presumption of difference) or based on wrong conception and misunderstanding and hence classical physics cannot be blamed for being unable to justify this equivalence which is not supposed to exist in the first place (since we have a single concept of mass and not two concepts of gravitational mass and inertial mass).[21]

In this context, we may make an analogy with charge in physics where no one thinks it is necessary to differentiate between "electric charge" (i.e. charge involved in electric laws) and "magnetic charge" (i.e. charge involved in magnetic laws).[22] So, should we need an equivalence principle for charge, similar to that of mass, to bring consistency to the electromagnetic laws?

We should finally remark that the measurement of G historically (as conducted for instance by Cavendish) was essentially based on the presumption of identicality of inertial mass and gravitational mass and this indicates the deep intuitive feeling of the existence of a single concept of mass (rather than two) in classical physics. This similarly applies to other historical experiments and investigations of classical physicists which are generally based on the (potentially implicit) presumption of the unique nature of mass (rather than being inertial and gravitational).

2. Distinguish between the different forms of the principle of equivalence and the scientific evidence in their support.

   Answer: We may distinguish between three main forms:

   (**a**) The classical principle of equivalence whose essence is the equivalence (or rather identicality) between inertial mass and gravitational mass. The evidence for the classical form is conclusive (to the reported degree of accuracy). In fact, this should be a confirmation to the classical conception of mass as a single concept (as explained in exercise 1).

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[21] Despite our belief that the equivalence principle (as "equivalence principle" and not as identicality of inertial and gravitational mass) has no place in the framework of classical physics, we continue to use terms like "classical equivalence principle" in our reference to the above fact to avoid deviation from the literature and to facilitate the comparison with the equivalence principle of general relativity.

[22] The meaning of "magnetic charge" here is different from its meaning in the context of magnetic monopole (whose existence is hypothetical).

(**b**) The general relativistic weak form of the equivalence principle whose essence is the equivalence between the mechanical effects of gravity and the mechanical effects of acceleration. It may be claimed that the evidence for the weak form is substantial. However, the actual evidence is for the classical content of this principle (i.e. the equivalence of inertial mass and gravitational mass) and not for the theoretical content (i.e. the equivalence of the effects of gravity and acceleration which is different in essence and more general, moreover it is based on a presumed distinction between inertial mass and gravitational mass) and hence the evidence is actually for the classical rather than the weak equivalence principle (see § 9.12 and § 10.1.1). We note that many of the claimed evidence in support of general relativity are related to the weak form and hence in our view they are actually evidence for the classical equivalence principle. This means that they do not actually endorse general relativity which is based on the theoretical content of this principle. We should also note that all the evidence in support of the classical principle of equivalence are generally regarded as evidence in support of the general relativistic weak equivalence principle and this should be rejected because the two (i.e. classical and weak) are not identical. Therefore, in our view there is no evidence in support of the weak equivalence principle (as such).

(**c**) The general relativistic strong form of the equivalence principle whose essence is the equivalence between the physical effects of gravity and the physical effects of acceleration.[23] There is no actual evidence for the validity of the strong form despite the alleged justification of this form by theoretical arguments. In fact, we may even claim that there is evidence in support of its invalidity (see § 9.12 and § 10.1.1). We note that the most important implications and consequences of general relativity (mainly as a "General Theory") are related to the strong form.

3. What is the difference between the weak and strong forms of the equivalence principle of general relativity?
   Answer: As explained earlier, the weak form is about the mechanical effects of acceleration and gravity while the strong form is about their physical effects in general (noting that the strong form contains generalizations about the invariance of physical laws and the local application of special relativity).
4. Discuss briefly the importance of the strong equivalence principle for the theoretical framework of general relativity.
   Answer: While general relativity starts as a gravity theory it ends, thanks to the strong equivalence principle, as a "General Theory" since it claims that all physical systems in local gravitational frames (and equivalent accelerating frames) behave essentially in the same way as in inertial frames of the Minkowski spacetime (where general coordinate transformations account for the extra effects of gravity and acceleration). In fact, the strong equivalence principle is what makes general relativity general relativity since this principle transforms the theory from being a mere gravity theory to a "General Theory" or general relativity.
5. Discuss the significance of the weak and strong forms of the equivalence principle and their relation to the nature of general relativity.
   Answer: In our view, the weak form is mainly related to general relativity as a gravity theory while the strong form is mainly related to general relativity as a "General Theory". This is inline with our belief that the strong form is in essence a generalization to the invariance principle (mainly of special relativity noting that this principle is represented by the first postulate, i.e. the principle of relativity).
   Note: a matter related to this discussion is the relation between the invariance principle and the equivalence principle (particularly its strong form). In this context, it may be claimed that the strong form of the equivalence principle requires or implies the invariance of the essence of the physical laws while the invariance principle of general relativity requires the invariance of the form of the physical laws by being cast in a certain form. However, this should depend on the definition and meaning of

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[23] We may say (inline with our previous insight or assertion): "whose essence is the equivalence of all frames of reference in their validity for formulating the laws of physics" noting that this sounds like the principle of invariance (or the "principle of relativity"). In fact, we believe that the two are very similar if not identical. This may be inline with the view of some relativists that "The Principle of General Covariance is a mathematical statement of the Equivalence Principle" (see Ryder in the References) which seems in accord with the *Principle of General Covariance* as stated by Weinberg in an early quote (see Problem 3).

these principles (as well as other factors) which are subject to different opinions and interpretations. Also, see exercise 26.

6. Summarize your main observations about the strong equivalence principle.
   Answer: We note the following:
   • The spacetime of general relativity is locally flat and hence special relativity applies locally. In other words, the laws of physics equally apply in all frames of reference at local level (see the upcoming note). This may be rephrased as: the laws of physics take the same form in all reference frames at local level.[24] So, the invariance principle is contained inside the strong equivalence principle and hence in a sense the strong form is an extension to the first postulate of special relativity (i.e. the invariance of physical laws) because this invariance is extended to include all laws and all frames.[25] For example, Maxwell's equations, which apply in the Minkowski spacetime of special relativity, will equally apply in the curved spacetime of general relativity at local level and hence these equations (in their forms and in the embedded universal constants) are locally valid in any frame (noting that they apply in their special relativistic form in local inertial frames, and they apply in their modified form in other frames that are transformed to from the local inertial frames).
   • The weak equivalence principle may be regarded as a special case of the strong equivalence principle. This should endorse our view that the weak form of the equivalence principle is not the same as the classical form due to the embedded theoretical component in the strong form and hence in the weak form.
   • There is no actual evidence in support of the general validity of the strong equivalence principle (we can even claim that there is evidence against it). In fact, we do not believe even in the existence of actual evidence in support of the weak equivalence principle (as such) because as indicated earlier the actual evidence is for the classical component that is embedded in the weak form and not for the additional theoretical component of the weak form.
   Note: we may say that the local flatness of the spacetime of general relativity ensures the local applicability of special relativity (since in the relativity theories the essence of the laws of physics should depend on the geometry of spacetime), while the availability of a local inertial frame (i.e. freely falling frame in which special relativity applies) plus the availability of valid coordinate transformations (i.e. from this local inertial frame to other local frames) ensures the form invariance of the transformed laws across the local frames (i.e. from the local inertial frame to the corresponding local gravitational or accelerating frames). Accordingly, the effect of gravity and acceleration on any law in any local frame should be accounted for by the transformation from the local inertial frame to that frame.
   In fact, this should reflect a general relativistic doctrine that the laws are essentially the same in any spacetime and in any frame (where these laws are essentially represented by the laws of special relativity). However, they take their "flat" special relativistic form in the inertial frames (which are potentially local realized by free fall) and they take their "curved" general relativistic form in other frames. So, the "curved" doctrine of general relativity seems to extend to all laws of physics and not restricted to gravity (although there is difference in the meaning and significance). Moreover, the effect of non-inertiality (whether acceleration or gravity) should be a matter of transformation across frames of reference and hence it seems to be physically void.
7. The equivalence principle of general relativity may be seen as a bridge between geometry and physics. How?
   Answer: The principle of equivalence links inertia (which is closely related to the geometry of spacetime) to gravity (which essentially is a physical agent) and hence from this perspective the principle can be seen as a bridge between the geometry of spacetime and its physics.
8. The equivalence principle of general relativity may be seen as having two sides (or meanings): static and dynamic. How?

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[24] It is claimed in the literature that the strong equivalence principle should also imply that the values of dimensionless physical constants should also be identical in all frames. In other words, the invariance should include value invariance as well as form invariance.

[25] In fact, "all laws" should have already been assumed in the invariance of special relativity (with gravity being excluded by the absence of any source of gravitation in the spacetime) so we add this to emphasize the point.

Answer: In its static side the equivalence principle means that a stationary frame in a gravitational field is equivalent to accelerating frame in free space. In its dynamic side the equivalence principle means that a freely falling frame in a gravitational field is equivalent to an inertial frame in free space.

9. Interpret the equivalence principle of general relativity in the light of the locality of frames and hence determine the role of general relativity and the procedure that should be followed in its application.
Answer: The equivalence principle (plus certain assumptions and known facts) implies that the curved spacetime of general relativity can be split into patches of approximately flat regions and hence it can be coordinated by a network of connected local inertial frames where each one of these frames is in a state of gravitational free fall and hence it is subject to the rules of special relativity. The role of general relativity (as a gravity theory) is to provide a description of the geometry of this curved spacetime through the correlation between the spacetime metric (which represents the geometry) and the mass-energy distribution in the spacetime (which represents the physics). This correlation is the essence of the Field Equation (see § 3.2). So, the procedure is that we first determine the geometry of spacetime using the Field Equation of general relativity, we then divide the spacetime into locally flat regions, we then apply the equivalence principle at each one of these regions to conclude that each freely falling frame in one of these regions is effectively inertial (since the free fall acceleration annuls the effect of gravity),[26] and we finally apply special relativity in each one of these regions since they are "flat" and coordinated by "inertial" frames (as it is the case with the flat spacetime of special relativity and its inertial frames). The effect of gravity (i.e. in non freely falling frames that correspond to the freely falling frames) will then be obtained by finding a general coordinate transformation from the freely falling frame to the corresponding gravitational frame.
10. Examine the significance of the locality restriction in the statement of the equivalence principle.
Answer: First, it is about being confined (i.e. local as opposite to global) in spacetime and hence it should be temporally, as well as spatially, local. Second, it depends on the required level of accuracy and this depends on the nature (e.g. degree of deviation from uniformity) and intensity of the gravitational field and the objectives of the experiment. So, the locality condition may be satisfied in a certain patch of spacetime if we seek a certain level of accuracy (e.g. 10^{-2}) but it may not be satisfied if we seek a higher level of accuracy (e.g. 10^{-5}).
11. Analyze the condition "sufficiently small region of spacetime" in the statement of the equivalence principle.
Answer: The apparent purpose of this condition is to eliminate the effects of gravitational gradients and the resulting tidal forces. However, we may argue that this condition cannot fulfill this objective completely because tidal forces do not vanish in small regions of spacetime although they may "vanish from observation" due to the limitation on the precision and resolution of physical experiments. In other words, we may eliminate tidal forces practically but not truly.
To sum up, in the presumed equivalence between gravity and acceleration (according to the equivalence principle of general relativity) we have two issues. The first is the equivalence between uniform gravity (where the above condition supposedly ensures the uniformity) and uniform acceleration, while the second is the tidal forces which are a distinctive feature of gravity. So, even if the presumed equivalence between gravity and acceleration can be established and justified theoretically and experimentally with regard to the first issue, it cannot be established and justified with regard to the second issue.
12. Analyze a freely falling frame in a gravitational field.
Answer: If we restrict our attention to a sufficiently small region of space and sufficiently small period of time (i.e. locality in spacetime) then we will observe that in a freely falling frame Newton's laws of motion hold true.[27] For example, an object subjected to a force in this frame will accelerate

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[26] This should justify the imposition of the locality condition since the acceleration caused by the free fall will not annul the effect of gravity if the locality condition is violated because in this case the effect of gravitational field gradients and tidal forces is not negligible and hence it cannot be ignored (noting that uniform acceleration can annul the "uniform component" of gravity but not its "non-uniform component" which is represented by the gravitational gradients and tidal forces).

[27] We note that Newton's laws of motion are example of the laws of not only classical mechanics but even Lorentz mechanics since according to this mechanics these laws (with some modifications) apply in inertial frames (noting that freely falling frame is locally inertial according to the equivalence principle).

according to Newton's second law while an object that is free of such a force will keep its state of rest or uniform motion. However, if the locality restriction is violated then we will observe that these laws do not hold true. For example, if we observe two objects that are at rest in this frame then over a sufficiently large spacetime interval we may see them converging toward each other horizontally (or laterally) or diverging from each other vertically (or radially) due to the gravitational field gradient and the resulting tidal force. In brief, this frame is inertial when the locality condition is satisfied and non-inertial when the locality condition is violated, i.e. it is locally inertial frame and globally non-inertial frame.

13. Show formally, using the classical equivalence principle (and assuming locality), that a freely falling frame in a gravitational field is effectively inertial (i.e. a frame in which Newton's laws of motion are valid).
Answer: Let for example have a test particle that is freely falling in the vicinity of the surface of the Earth where the particle has an inertial mass m_i and a gravitational mass m_g. Now, let the particle as seen from the Earth rest frame (assuming it is inertial) be subjected to a non-gravitational force $\mathbf{f}_{ng} = m_i \mathbf{a}_{ng}$ where $\mathbf{a}_{ng}$ is the acceleration caused by the non-gravitational force $\mathbf{f}_{ng}$. Accordingly, in the Earth rest frame the total non-fictitious force $\mathbf{f}_{nf}$ to which the particle is subjected is: $\mathbf{f}_{nf} = m_g \mathbf{g} + m_i \mathbf{a}_{ng}$ where $\mathbf{g}$ is the gravitational field in the vicinity of the Earth surface.[28] Now, in the freely falling frame of this surrounding we also have a fictitious force $\mathbf{f}_f = -m_i \mathbf{g}$ due to the gravitational acceleration. Accordingly, in the freely falling frame the total force (i.e. non-fictitious and fictitious) to which the particle is subjected is: $\mathbf{f}_{nf} + \mathbf{f}_f = m_g \mathbf{g} + m_i \mathbf{a}_{ng} - m_i \mathbf{g}$. So, if $m_g = m_i = m$ (according to the classical equivalence principle) then in the freely falling frame the total force to which the particle is subjected is: $\mathbf{f}_{nf} + \mathbf{f}_f = m\mathbf{g} + m\mathbf{a}_{ng} - m\mathbf{g} = m\mathbf{a}_{ng}$. This means that in this frame the particle follows Newton's second law (and hence it should also follow Newton's first law which is a special case of Newton's second law and should similarly follow Newton's third law). Therefore, a freely falling frame in a gravitational field is effectively inertial.
Note: the use of $\mathbf{g}$ as gravitational field and as gravitational acceleration may also need justification by the classical equivalence principle and hence the above argument can be questioned. In fact, it may be questioned from other aspects as well.
14. Analyze a frame that is uniformly accelerating in the absence of any gravitational field.
Answer: If we follow a similar line of reasoning to that of exercise 12, then we can say: this frame will look locally like a gravitational frame, e.g. a free object will behave like a gravitationally falling object in this frame. However, globally it is not a gravitational frame because its non-gravitational nature can be exposed when the locality condition is violated. For example, two objects that are falling in this frame will not converge or diverge because the "acceleration field" is uniform while any realistic gravitational field is non-uniform since it converges toward the center of the gravity source (or gravitating body) and it has radial gradient.
15. Discuss and assess the equivalence principle in the context of free fall situation.
Answer: We note the following:
• Since gravity is equivalent to acceleration of suitable magnitude in the opposite direction to the gravitational field, free fall in a gravitational field (which means acceleration of suitable magnitude in the same direction as the gravitational field) should cancel the effect of gravity (although gravity is still there) and hence a free fall situation is effectively equivalent to a gravity-free state in spacetime.
• Referring to exercise 11, in free fall situation tidal forces do not vanish and hence the equivalence between gravity and acceleration is not complete. The practical elimination of these forces by imposing the locality condition does not change the nature of gravity and its distinction from acceleration.
16. Derive a formula for the gravitational field gradient (which estimates tidal forces) along the radial direction in the neighborhood of a gravitating object using the classical formalism of gravity (see § 1.5).
Answer: From Newton's gravity law, the strength of gravitational field g (i.e. the magnitude of

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[28] The non-fictitious nature of the force is due to the inertiality of the frame of observation.

gravitational force f per unit mass m) of a gravitating object with mass M is given by:

$$g \equiv \frac{f}{m} = \frac{GM}{r^2} \tag{3}$$

where m is the mass of the gravitated object, G is the gravitational constant and r is the radial distance between the centers of the two objects. Hence, the gravitational field gradient[29] along the radial direction is given by:

$$\frac{dg}{dr} = -\frac{2GM}{r^3} \tag{4}$$

Accordingly, the gravitational field gradient (and hence the tidal force) is directly proportional to the mass of the gravitating object and inversely proportional to the cube of the radial distance. Hence, the gravitational field gradient (and the tidal force) will be stronger near more massive objects and at locations closer to the center of the gravitating object.[30] As we see, the magnitude of the gravitational field gradient increases rapidly as the radial distance decreases due to the inverse cubic relation.

17. Calculate the tidal force f_t (i.e. force per unit mass)[31] along the radial direction over a distance $d = 1$ nm in the vicinity of the Earth (**a**) at its surface and (**b**) at distance $2R$ from its surface (where R is its radius). Comment on the results.
Answer: We use the result of exercise 16, that is:
(**a**) At its surface we have $r = R$ and hence:

$$\begin{aligned} f_t &\simeq \frac{dg}{dr} \times d = -\frac{2GM}{r^3} \times d \simeq -\frac{2 \times 6.674 \times 10^{-11} \times 5.972 \times 10^{24} \times 10^{-9}}{\left(6.371 \times 10^6\right)^3} \\ &\simeq -3.083 \times 10^{-15}\,\text{N/kg} \end{aligned}$$

where the minus sign indicates that the force is stretching (i.e. it is stronger at the near side) rather than compressing.
(**b**) At distance $2R$ from its surface we have $r = 3R$ and hence the result of part (a) should be scaled by a factor of $(1/3)^3 = 1/27$, i.e. $f_t \simeq -1.142 \times 10^{-16}\,\text{N/kg}$.
Comment: in both cases the tidal force is negligible.
Note: strictly, the gravitational field gradient (and hence the tidal force) belongs to a point since it depends on the position. Therefore, f_t in the above calculations (and in the next exercise) should represent an average and hence we use the approximation symbol. We should also note that the approximation can be justified from a practical perspective by the very tiny size of the distance d since the tidal force does not vary substantially over this distance and hence it is like a constant.

18. Estimate the tidal force f_t (i.e. force per unit mass) along the radial direction over a distance $d = 1$ nm in the vicinity of a black hole of 1 Earth mass (**a**) at the event horizon[32] and (**b**) at distance $2R_S$ (where R_S is its Schwarzschild radius) from the event horizon. Comment on the results.
Answer: We use the result of exercise 16 as an approximation (see the upcoming note), that is:
(**a**) At the event horizon we have $r = R_S$ and hence:

$$\begin{aligned} f_t &\simeq \frac{dg}{dr} \times d = -\frac{2GM}{R_S^3} \times d = -\frac{2GMd}{\left(2GM/c^2\right)^3} = -\frac{c^6 d}{4G^2M^2} \\ &\simeq -\frac{\left(3 \times 10^8\right)^6 \times 10^{-9}}{4 \times \left(6.674 \times 10^{-11}\right)^2 \times \left(5.972 \times 10^{24}\right)^2} \simeq -1.147 \times 10^{12}\,\text{N/kg} \end{aligned}$$

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[29] In fact, this is the magnitude of the gravitational field gradient with the minus sign indicating the decreasing nature of force as a function of r. Its direction is along the radial orientation.

[30] We remind the reader that the mass M in the above formulation should be contained within the radial distance r. In other words, the gravitated object is outside the gravitating object.

[31] What is actually required is its magnitude with the sign indicating its nature.

[32] The event horizon of a black hole is an imaginary sphere whose center is the singularity (or center) of the black hole and whose radius is the Schwarzschild radius ($R_S = \frac{2GM}{c^2}$) of the black hole (see § 8.7).

(**b**) At distance $2R_S$ from the event horizon we have $r = 3R_S$ and hence the result of part (a) should be scaled by a factor of $(1/3)^3 = 1/27$, i.e. $f_t \simeq -4.249 \times 10^{10}$ N/kg.
Comment: in both cases the tidal force is enormous. Moreover, there is a huge difference between the results of this exercise and the results of exercise 17 due to the fact that the gravitational fields generated by black holes are much stronger than the gravitational fields generated by ordinary objects and this also applies to the tidal forces.[33] However, the correspondence between the two cases is questionable because we are comparing the tidal force at the surface of the Earth (and its multiple) to the tidal force at the event horizon (and its multiple) where the difference in the size of the radial distance from the center of the gravitating object in the two cases is huge. Yes, we may see this as a comparison between the force at the surface of the Earth and the force at the "surface" of the black hole (which is the event horizon).
Note: the validity of the use of the result of exercise 16 as a legitimate approximation may be questioned because the gravitational field of the black hole is very strong and hence the classical formulation is not valid (see § 5). However, this may be seen as a rough estimation of the tidal force and hence the use of the classical formulation could be justified at least from a qualitative perspective. In fact, a similar equation to the above classical equation for the "field gradient" (or rather what is general-relativistically equivalent to the classical "field gradient") can be obtained from general relativity although the interpretation and significance (e.g. the meaning of coordinates) of the two formulations are not the same (the reader is referred to § 6 for the meaning and interpretation of coordinates in general relativity).

19. Assess the validity of the equivalence principle (and hence the validity of general relativity which is based on this principle) in very strong gravitational fields.
Answer: The essence of the equivalence principle is that gravity and acceleration are equivalent at local (or infinitesimal) level.[34] Now, gravitational field gradients (which determine tidal forces) are given by $\frac{dg}{dr} = -\frac{2GM}{r^3}$ (see exercises 16 and 18) and hence they are directly proportional to the gravitating mass and inversely proportional to the cube of radial distance. So, in very strong gravitational fields generated by compact objects like black holes[35] the gravitational field gradients and the tidal forces become substantial in the vicinity of these objects even at local (or infinitesimal) level (as seen in exercise 18) and hence they cannot be ignored because even locally (or infinitesimally) there should be physical restrictions on the system resolution, i.e. we are not free to reduce the size of the system as we wish to make the gravitational field gradients and tidal forces negligible. So, even if conceptually and theoretically we can make gravity equivalent to acceleration, we may not be able to justify this physically and practically. This should put a question mark on the validity of general relativity itself (which is based on the equivalence principle) at least in very strong gravitational fields (which should have strong gradients). In fact, this challenge should target general relativity both as a gravity theory and as a "General Theory".
Note 1: as indicated earlier, some general relativists reject the equivalence principle and consider it as motivation rather than foundation for general relativity. According to this view, the invalidity of the equivalence principle does not affect the validity of general relativity. However, we believe that even if this is the case with the weak form of the equivalence principle it is not the case with the strong form (which contains elements of invariance and generalizations that are missing in the weak from) and hence the collapse of the equivalence principle should lead to the collapse of general relativity or at least some of its generalizations. In other words, it should lead to the collapse of general relativity as a "General Theory" even though it may still be valid as a gravity theory. This should be inline with

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[33] In fact, this is because of the compactness of black holes and the dependency of the gravitational field gradient on the inverse cubic distance.

[34] The justification for "infinitesimal" is that the locality condition should at least work at this limit in ordinary physical systems. In fact, in most cases the "infinitesimal" condition is stronger than the "local" condition from a practical perspective.

[35] This is not as general as it sounds. In fact, it applies to certain types of black hole as this can be concluded from the correlation between the gravitational field gradient and the mass (which is an inverse square relation as seen in exercise 18 above and in exercise 20 of § 8.7). However, we refer to those cases in which our argument applies and this does not affect its validity or generality.

our view that the weak form is mainly related to general relativity as a gravity theory while the strong form is mainly related to general relativity as a "General Theory" (see exercise 5). So in brief, we perhaps can have a valid general relativistic gravity theory without the weak equivalence principle but we cannot have a valid general relativistic "General Theory" without the strong equivalence principle because the "General Theory" is the essence of the strong equivalence principle and hence its collapse should lead to its collapse.

Note 2: although we are using the classical formulation for the gravitational field gradient in the above argument, the argument should still be justified at least from a qualitative perspective, as indicated in the note of exercise 18.

20. Are tidal forces restricted to the radial direction?
Answer: No. There should also be tidal forces along the lateral (or circumferential or transversal) direction. The effect of lateral tidal forces becomes prominent in highly non-uniform gravitational fields (e.g. fields generated by a complex formation of matter rather than by a spherically symmetric object) where the lateral component of the field gradient is tangible. The effect also becomes prominent during the side-by-side movement of gravitated objects in free fall due to the convergence of the gravitational field as the objects move toward the gravitating object, and hence two freely falling objects side-by-side will feel attracted to each other and accelerated toward each other. We note that the lateral acceleration due to the movement in non-uniform gravitational field may be labeled specifically as "geodesic deviation" (which represents a general relativistic paradigm; see § 7.13) although this term should be used in a more general sense that includes this case.

21. Make a brief assessment to the weak and strong forms of the equivalence principle.
Answer: We may criticize the weak and strong forms by the existence of tidal forces in gravitational fields, as discussed in the previous exercises. The strong form may also be criticized by the fact that such a generalization lacks experimental support.[36] In fact, we may even claim the existence of experimental evidence against the general validity of the strong form of the equivalence principle. The strong form also lacks logical foundation because even if we accept that gravity and acceleration are equivalent in their direct mechanical effects, it is not obvious that they should be equivalent in every aspect and hence any physical system will behave equally in gravitational frames as in accelerating frames and both should be locally subject to the rules of special relativity.

22. Use the existence of tidal forces in gravitational systems as a basis for a potentially fundamental challenge to the strong equivalence principle.
Answer: The strong equivalence principle can be challenged by the possibility of the dependence of physical laws (or at least some of them) on the existence and non-existence of gravitational field gradients and tidal forces (regardless of being detected or not detected by an observer). So, the apparently-intuitive theoretical arguments in support of the equivalence principle may establish the weak form but not the strong form of the equivalence principle. In fact, using the locality as a basis for having good quantitative approximation may be valid with respect to the weak form but it may not be valid for establishing the strong form since the existence and non-existence of gravitational field gradients and tidal forces may result in a qualitative difference between gravity and acceleration and this qualitative difference may invalidate the strong form of the equivalence principle altogether, i.e. the physical system in which tidal forces do exist may be totally different in its behavior from a similar physical system in which tidal forces do not exist and hence the argument of the locality-based approximation will become entirely irrelevant because the difference is qualitative and not quantitative.

23. Discuss the issue of decelerating frames (as a subset of accelerating frames in its general sense) and to which gravity they should correspond according to the equivalence principle. Also, discuss the implication of this on the existence of absolute frame.
Answer: Decelerating frames should correspond to repelling (rather than attracting) gravity (which we may call negative gravity)[37] and this should put more question marks on the sensibility and generality (and even validity) of the equivalence principle. However, this should depend on a clear distinction

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[36] Even the weak form (as such) lacks experimental support because the experimental support is for its classical content.

[37] Considering this as repelling or negative gravity rather than changing the position of the gravity source is to ensure consistency. This should be more obvious when we consider a single frame which alternately accelerates and decelerates.

between acceleration and deceleration which can be easily established due to the undeniable existence of inertial frames (or absolute frame if we believe in absolute frame) as a distinguishing agent between acceleration and deceleration. Anyway, from a practical perspective the distinction is obvious due to the difference in the observed physical effects in the two cases, i.e. pulling in one direction in acceleration and pushing in the opposite direction in deceleration.

24. To what type of gravitation rotating frames correspond?
    Answer: In fact, this can be regarded as another source of limitation in the equivalence principle because we have a type of acceleration (i.e. rotation) that has no gravitational counterpart and this is a clear indication that gravity and acceleration are physically different. We should also remind the reader of other restrictions on the equivalence principle (such as uniformity of acceleration) which should diminish the value of the equivalence principle.
    We should note that any attempt to restrict acceleration in the statement of the equivalence principle to a certain type (apart from being uniform which is stated explicitly)[38] should be rejected because the ultimate purpose of the equivalence principle is to extend the validity and applicability of physical laws from inertial frames to all types of frame (i.e. accelerating, decelerating, rotating and gravitating) by unifying these frames through the equivalence principle and this cannot be achieved if acceleration (in its general sense) is restricted to a certain type (e.g. translational).
25. Try to challenge our criticism to the equivalence principle for its failure to include decelerating and rotating frames (as well as non-uniformly accelerating of all types) and assess this challenge.
    Answer: Our criticism may be challenged by questioning the necessity of including other types of non-inertial frame (i.e. other than translationally uniformly accelerating) for the validity of the equivalence principle. This is because what is needed for establishing the equivalence principle is the equivalence between uniform gravity (whose uniformity is established by the locality condition) and uniform translational acceleration while the status of other types of non-inertial frames should be determined by a transformation between the gravitational frame and these types of frame, and hence these types of frame are not involved in the equivalence principle. In other words, what is required for the validity of the equivalence principle is only the validity of the equivalence between gravitational frame and translationally uniformly accelerating frame because we are considering local patches of spacetime (which are effectively flat) coordinated by freely falling frames (which are effectively inertial) and hence what is required for the validity of the equivalence principle is only the equivalence between "local" gravity (which is uniform due to locality) and translational uniform acceleration. Any physical effect that originates from the physical status of these types of frame (e.g. being non-uniform or being decelerating or being rotating) will then be accounted for by a transformation between the gravitational frame and these frames. However, this challenge can be refuted by the fact that we then need another principle to justify the transformation between the gravitational frame and these frames. In other words, we need anyway an extension to the equivalence principle to include the equivalence between gravity and these types of non-inertiality to justify this transformation. This is because the mere existence of a formal transformation between gravitational frame and these frames does not guarantee the physical transformation of the actual physical effect that is based on the status of these frames (e.g. being rotating) and hence the equivalence principle (with its restriction to uniform translational acceleration) is not sufficient to determine the physical situation in these frames and fill the gaps.
26. What is the relation between the invariance principle and the equivalence principle in general relativity?
    Answer: We can claim that an element of invariance is embedded in the strong form of the equivalence principle. This is because the strong form requires that all physical laws (i.e. in their invariant form) should equally apply in all frames at local level and hence in a certain sense the invariance of physical laws across all frames of reference (according to the invariance principle) is contained in the strong form of the equivalence principle.[39] However, we may also claim that the invariance in the strong equivalence principle is basically related to the essence of the physical laws while the invariance in the invariance principle is related to the form of the physical laws (see exercise 5).

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[38] In fact, the condition of uniformity should also impose a limitation on the equivalence principle due to the ambiguity of the status of non-uniformly accelerating frames.

[39] This should also be inline with the *Principle of General Covariance* (see Problems).

We should note that in the literature of general relativity there are different opinions about this relation,[40] and this is just one of the many conflicts and controversies that infest the relativity theories. We outlined our opinion already and it will be clarified further in the future. We should also draw the attention to the *Principle of General Covariance* which was discussed in the Problems and will be discussed further in the next exercises (and later on).

27. Outline the technical meaning of the *Principle of General Covariance*. What is the procedure for implementing this *Principle* and what is the logic behind this procedure?

Answer: According to this *Principle*, any law that is valid in the flat spacetime of special relativity should also be valid in the spacetime of general relativity if there is a valid coordinate transformation that can transform this law invariantly from the flat spacetime to the curved spacetime (where the curvature reflects the effect of gravity). The form of the law in the curved spacetime will then be the transformed form of the law to the (local) gravitational frame.

The procedure for implementing this *Principle* is to put the special relativistic law in a tensorial form (to keep its form-invariance under general coordinate transformations) followed by transforming from the freely falling local frame to the gravitational frame. The logic behind this procedure is that the law is valid in special relativity and hence by the equivalence principle it should be valid in a freely falling local frame (since it is locally inertial frame coordinating flat patch of spacetime), so any effect of gravity on the law will be taken into consideration by the general transformation from the freely falling frame to the gravitational frame.

28. Assess the *Principle of General Covariance*.

Answer: We note the following:

• It lacks physical evidence (experimental and observational). In fact, we may even claim the existence of physical evidence against this *Principle* due to its link to the strong form of the equivalence principle (see for example § 10.1.1).

• It lacks logical foundation and justification because the existence of a valid formal transformation from a freely falling local frame to a gravitational frame does not guarantee the physical transformation of the actual physical effect from one frame to the other. In other words, the mathematical transformation does not imply physical transformation of the actual effect (i.e. the incorporation of the effect of gravity) in the real world because the mathematics of transformation may not reflect the physics of transformation.

• This *Principle* requires the existence of a valid general transformation between the freely falling local frame and the gravitational frame. However, it is not obvious that such a transformation does exist (at least practically) in all cases and circumstances.

• The restrictions and gaps in the equivalence principle should also apply to this *Principle* and hence this *Principle* does not apply in many legitimate physical situations. These restrictions and gaps include limitations on the type of frame to which we transform, e.g. non-uniformly accelerating, decelerating, and rotating frames. For example, the acceleration in the statement of the equivalence principle is restricted to be uniform and hence this *Principle* cannot be applied to non-uniformly accelerating frames.

Note: the last point may be challenged by questioning the validity of the criticism of the equivalence principle by being restricted to certain types of frame. This is because these types of frame are what we transform to and they are not involved in the equivalence principle. In other words, what is required to the validity of the equivalence principle (and hence the validity of the *Principle of General Covariance*) is only the validity of the equivalence between gravitational frame and translationally uniformly accelerating frame because we are considering local patches of spacetime coordinated by freely falling inertial frames and hence what is required for the validity of the equivalence principle is only the equivalence between "local" gravity (which is uniform due to locality) and uniform acceleration. Any physical effect that originates from the physical status of these types of frame (e.g. being non-uniformly accelerating or being rotating) will then be accounted for by the transformation between the gravitational frame and these frames. However, this challenge can be refuted by the fact that we then need another principle to justify the transformation between the gravitational frame and these frames. In other words, we still need an extension to the equivalence principle to include the equivalence between gravity and these types of non-inertiality to justify this transformation. Also see exercise 25.

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[40] In fact, this difference in opinion is largely based on the difference in opinion about the meaning and interpretation of "invariance" (e.g. being generic or technical or *general covariance*) as well as the difference in opinion about the equivalence principle.

29. Justify the following statement using the invariance and equivalence principles: "physical objects that are not under the influence of non-gravitational forces follow geodesic trajectories".

    Answer: According to the equivalence principle gravity is not a force (since it is a geometric attribute of spacetime due to its equivalence to acceleration) and hence such physical objects are free objects. Therefore, according to (a rather extended form of) the invariance principle such objects should follow geodesic trajectories because in the flat spacetime of special relativity free objects follow straight trajectories (which are the geodesics of the flat spacetime) and hence they should follow the same type of trajectories (i.e. geodesics) in the curved spacetime of general relativity.

30. Identify the potentially different types of invariance as embedded in the general principles of general relativity.

    Answer: A distinction may be made between the invariance of the essence (or content) of the physical laws (which may be claimed to be the subject of the strong equivalence principle) and the invariance of the form of the physical laws (which may be claimed to be the subject of the invariance principle although the significance of this principle is commonly extended to include even the invariance of the essence) noting that these two types may be combined in a single principle (e.g. the *Principle of General Covariance* as a potential candidate for this combination).

    Note: as indicated earlier, we believe that while the invariance of the essence of physical laws should be a necessity the invariance of the form has mainly conventional value (see exercise 8 of § 1.8.1). However, this does not mean the acceptance of the strong equivalence principle (even if it is really about the invariance of the essence of the laws) due to the difference in the perspectives and conditions, i.e. in our case the difference between the systems should be conventional for the essence to be invariant while in the case of the strong equivalence principle the difference could be (and usually is) real and physical and hence the invariance of essence is not obvious (at the best).

31. Discuss how the principle of relativity in special relativity is extended by the equivalence principle to the "principle of relativity" in general relativity.[41]

    Answer: The essence of the principle of relativity in special relativity is that the laws of physics are the same in all inertial frames. By the equivalence principle of general relativity, the principle of relativity becomes: at local level the laws of physics are the same in all freely falling frames (since they are locally inertial coordinating flat patches of spacetime and hence they should be subject to the laws of special relativity).[42] The extension to all types of frame (whether freely falling or not) will then be justified by the presumed existence of valid transformations from the freely falling frames to the other types of frames. The validity of this extension should rely on the (non-obvious) claim that the formal transformations will incorporate the actual physical effects that differentiate the other types of frames from the freely falling frames.

1.8.3 The Principle of Correspondence

1. Define and discuss the correspondence principle.

    Answer: The correspondence principle is a general epistemological principle that applies to any novel theory that is supposed to replace a corresponding theory in classical physics and hence it departs from some classical predictions. So, the correspondence principle applies to quantum mechanics, Lorentz mechanics, general relativity and any other new theory. This principle requires that any novel theory should converge to the corresponding classical theory in the domain of validity of the classical theory and hence quantum mechanics should converge to classical mechanics in quantum systems with large quantum number, Lorentz mechanics should converge to classical mechanics at low speed, and general relativity should converge to classical gravity in time independent weak gravitational fields with speeds much lower than c (refer to § 5). The justification of the correspondence principle is that classical physics is a well tested theory in its domain of validity and hence no theory should contradict classical physics in this domain.

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[41] In fact, this question is about the relation between the principle of relativity and the general principles of general relativity. The restriction to special relativity is because there is no such explicit "principle of relativity" in general relativity (although its essence is embedded in the framework of the theory).

[42] As explained earlier, being locally inertial is the result of the equivalence between gravity and acceleration since gravitational effects will be canceled by acceleration effects (and vice versa) and hence the freely falling frame will be free of the effects of gravity and acceleration and hence it is effectively inertial.

2. Is the correspondence principle restricted to physics?
   Answer: No. In fact, the correspondence principle in its most general form means that any new theory (whether physical or not) should agree with the verified predictions of the corresponding old theory (whether classical physics or something else). This is justified by the nature of the correspondence principle as an epistemological rule that originates from the principles of reality and truth and hence there is no restriction on its general validity. In fact, the correspondence principle should be generalized further to ensure the consistency of any new theory or observation with any old theory and observation in their domain of validity.
3. Discuss the applicability of the correspondence principle in the alleged convergence of general relativity to special relativity at local level.
   Answer: We do not believe that this convergence is natural and by the theory itself, but it is imposed externally by assuming the local validity of the formalism of special relativity in the spacetime of general relativity. In other words, the gravitational formalism of general relativity and the Lorentzian formalism of special relativity are linked together rather arbitrarily in general relativity as a "General Theory" without one of these formalisms being a special case of the other formalism. Accordingly, the correspondence principle (in the alleged convergence of general relativity to special relativity at local level) does not apply. This should also mean that any evidence in support of one of these theories belongs entirely to that theory and does not validate the other theory (see § 9.13).

1.8.4 Other Principles

1.9 Criteria of Scientific Theory

1. Can a creation theory be a scientific theory?
   Answer: In our opinion, no creation theory can be a scientific theory because it stands on the border between physical and non-physical (or supernatural) and hence it cannot be a physical theory in its entirety. We should remark that not being scientific does not mean being wrong. In fact, being right or wrong depends on the system of standards and values that we adopt in our judgment and this depends on the conventions, presumptions and topics (e.g. science, philosophy, religion, etc.). So, the issue of being right or wrong is more general and more complicated than the issue of being scientific or not.
2. Assess general relativity in the light of the above criteria.
   Answer: General relativity seemingly meets the first two criteria in general although there are many reservations on various aspects of the theory and its applications.[43] Anyway, its compliance with the third criterion is questionable due, for example, to the implication of dark energy or creation in some of its applications (see § 10.1.2 and § 10.1.3). Regarding the fourth criterion, the theory is generally compliant with the known facts, e.g. it agrees with the verified predictions of classical gravity.[44] Yes, some of the aspects and implications of general relativity require further verification (which is generally the case in any theory). As for the fifth criterion, the theory is generally compliant with this criterion although due to its highly theoretical nature there are many implications and consequences of the theory and its applications that are beyond the reach of the contemporary science (e.g. many details of the physics of black holes as well as most of relativistic cosmology). Also, the theory is not very practical to use and apply due to its exceptional complexity (see § 10.1.23). So, we can say that the theory is compliant with this criterion in most, but not all, details.

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[43] In fact, there are many inconsistencies and paradoxes in the theory and its applications (see for example § 10.1.20).

[44] In fact, this should be restricted to the "gravity theory" aspect due to the lack of evidence in support of the "General Theory" aspect (and possibly the existence of evidence against this aspect).

3. Give broad criteria for ideal scientific theory and assess the theories of modern physics from this perspective.
   Answer: In our view, the most important criteria in this regard are:
   • The theory should be as close as possible to common sense and our classical heritage and based on natural intuition.
   • It should be as simple as possible, i.e. it does not include unnecessary complications and twists.
   • It should emerge from observation and experiment rather than from pure theoretical speculations and mathematical modeling.
   • It should not be based on or include mathematical artifacts and theoretical fantasies and illusions, i.e. it must be sufficiently realistic.[45]
   • It should be compatible and harmonious in its methodology and style with other well established scientific theories so that it can be easily combined and merged with other theories and branches of science.
   Unfortunately, modern physics (including general relativity) is far from being ideal since unnecessary complications, departure from common sense and intuition, artifacts and fantasies (as well as many other bad things) are very common and they are seen as an advantage or even a necessity to science and its progress. So, the theories of modern physics (and general relativity in particular) are generally not ideal.
4. Clarify criterion (e) in the list that we stated in the text giving some examples.
   Answer: Criterion (e) means that any scientific theory should have implementable, testable and observable consequences. A theory that lacks practicality does not belong to science and hence it cannot be a scientific theory. An example of this is the string theory[46] (at least in its current state of development) where all (or almost all) of its predictions and consequences are beyond the reach of current science since they cannot be tested and verified by the contemporary scientific methods and techniques. Another example is the theory of Hawking radiation (at least with respect to stellar and super-massive black holes) where this alleged radiation cannot be verified (at least currently) by any available physical means. Criterion (e) also means that the theory must be usable in practice and hence it can be applied to obtain solutions to real problems within the domain of validity of the theory. Accordingly, a theory that is not usable (due to its immense complexity for instance) should not be considered a scientific theory.
   We should note that a theory may be partially practical and hence it should be considered partially scientific, i.e. it is scientific with respect to its practical aspects and non-scientific with respect to its non-practical aspects. In fact, most theories of modern physics (including general relativity and relativistic cosmology) can be classified as partially scientific since they contain many non-practical aspects.
5. Are the criteria for acceptable scientific theory (as stated in points a−e in the text) sufficient or/and necessary conditions (or not) for the truthfulness of a presumed scientific theory?
   Answer: All these criteria (individually and collectively) are necessary but not sufficient for the truthfulness of the theory because truthfulness requires actual physical evidence (i.e. from experiment and observation) for the validity of the theory. We note that even if criterion (d) means having actual physical evidence in support of the theory (rather than being just compatible and non-contradictory with other facts and verified theories) it is usually not sufficient for the acceptance of the theory because it is usually partial evidence.[47] Yes, these criteria combined should be sufficient for the provisional acceptance of the theory if there is enough physical evidence in its support and with no physical evidence against it.
   Note: truthfulness in this context should be interpreted practically (rather than idealistically) as providing correct physical predictions in all possible cases and circumstances to which the theory can apply. Truthfulness may also be interpreted as practical acceptance.

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[45] We note that some types and extent of fantasies and illusions can make the theory non-scientific (rather than non-ideal).

[46] Or rather: string theories.

[47] In fact, practically no theory can be proved to be correct in every single case of its application and in all circumstances with absolute accuracy and certainty.

6. Why metaphysical entities cannot enter in the formulation of a physical theory?
Answer: This is justified by the nature of physics and metaphysics which are different and have different domains of application and validity. Moreover, metaphysical premises and entities (e.g. the existence of deity) cannot be proved or disproved (at least by physical means) and hence they have no physical substance or value. In other words, they are physically nonsensical and meaningless.
7. Give more justification to the requirement of consistency with logic (i.e. criterion a).
Answer: Since one of the main objectives of any rational theory (especially scientific theory) should be the rationalization (or logicalization) of our observations, the consistency with logic should be an obvious requirement for the acceptance of any theory. Yes, sometimes we may be satisfied with finding practical rules that meet our practical needs only and hence we may not care about the rationality of the theory. However, such an approach is short sighted since it will eventually lead to contradictions and clashes even practically. Moreover, it is inconsistent with our nature as humans that have transcendental needs in addition to the basic practical and biological needs (and this indeed is what distinguishes humankind from other species of animal and largely justifies even the existence of science as an intellectual structure).

1.10 Reading Too Much in the Equations

1. Give some reasons for our rejection of over-mathematization of science.
Answer: For example:
• Mathematical models may be limited since any model is partially validated by a limited number of observations and hence its generality is not guaranteed. This means that if mathematical modeling leads observations it could lead to wrong conclusions and deviation from the correct path of development.
• Mathematical models and equations usually reflect only some aspects of the physical phenomena (even when they look general and comprehensive) and hence they are not necessarily correct in predicting all the consequences of their formality.
• Mathematics is a tool that is created by human brain using the fundamental principles of logic in association with abstraction, quantification and generalization. Accordingly, it is not guaranteed that Nature follows our mathematics in every detail even when there is some agreement between the two, i.e. mathematics as represented by our mathematical models and Nature as represented by our observations. In other words, the logic of human may not be identical to the "logic" of Nature even though the two logics are very similar since the logic of human is developed from an evolutionary process that is based on the fundamental principles and laws of Nature and hence our logic is a reflection in some sense to the "logic" of Nature. We may even claim that the presumption (which physics is generally based on) that Nature is a mathematized entity (i.e. its properties and behavior are regulated by the patterns and rules of mathematics) in every detail is not obvious, and hence Nature may violate the patterns and rules of our mathematics in some details.
• Generally, mathematical models are more subjective while observations are more objective and hence if mathematical modeling takes the lead it is very likely that science will be more susceptible to prejudice (intentional or non-intentional), humanization, personal convictions and stereotypes.
• Generally, mathematical models are valid in certain regimes but not in all regimes. For example, classical gravity models are valid in the weak gravitational regimes but not in the strong gravitational regimes. Similarly, general relativity could be valid in certain gravitational regimes (e.g. gravitational regimes of certain range of strength) and hence it cannot be extended automatically and without observational evidence to all regimes.

Chapter 2
Mathematical Background

2.1 Space, Coordinate System and Transformation

1. What is the relation between space, coordinate system and transformation?
   Answer: Coordinate system is a device used to identify points of space and describe its geometry, while transformation is a relation used to map one space onto another space (or map one coordinate system of a space onto another coordinate system of the space).
2. Characterize the spaces of special relativity and general relativity.
   Answer: The space of special relativity (i.e. Minkowski spacetime) is flat pseudo-Riemannian, while the space of general relativity is curved pseudo-Riemannian.
3. What is the difference between Riemannian space and pseudo-Riemannian space? Link this to the spaces of the relativity theories.
   Answer: Riemannian space and pseudo-Riemannian space are essentially the same but they are distinguished because of mathematical technicalities and conventions. According to these conventions, the quadratic form (which is the square of the line element of the space and it is based on the metric tensor) of Riemannian space is positive definite while the quadratic form of pseudo-Riemannian space is not positive definite and hence it can be zero or negative (as well as positive). Therefore, the spaces of special relativity and general relativity are pseudo-Riemannian because their quadratic form $(d\sigma)^2$ can be zero or negative. However, these mathematical technicalities are of little significance to physicists and hence the spaces of special relativity and general relativity are generally described as Riemannian. We should note that the space of special relativity is generally described as Minkowskian (as opposite to Riemannian or pseudo-Riemannian) to be more specific (i.e. Minkowskian space is a special type of pseudo-Riemannian space).
4. Should Riemannian space be curved?
   Answer: No. Riemannian space is more general than flat and curved and hence it includes both although it is common in the literature of relativity theories to use "Riemannian" to refer to curved spacetime and use "Minkowskian" to refer to flat spacetime.
5. List the main characteristics of Riemannian space.[48]
   Answer: The main characteristics are:
   - The geometry of the space is represented by a rank-2 symmetric tensor g_{ij}.
   - The square of the line element ds of the space is a homogeneous quadratic function in the differentials of the space coordinates (multiplied by the metric coefficients), i.e. it is of the form $(ds)^2 = g_{ij} dx^i dx^j$.[49]
   - The space is locally Euclidean (or flat) and hence it can be coordinated locally by a Cartesian (or pseudo-Cartesian) system.
6. What is the coordinate system of the spaces of special relativity and general relativity?
   Answer: The space of special relativity and general relativity is the 4D spacetime. Hence, the "coordinate system" of this space is the frame of reference that consists of one temporal coordinate and three spatial coordinates.
7. In the special and general theories of relativity, what transformation from one coordinate system to another coordinate system means?
   Answer: From the answer of the previous exercise, we conclude that it means transformation from one reference frame of spacetime to another reference frame. For example, in special relativity we

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[48] This question is about Riemannian as opposite to non-Riemannian and not as opposite to pseudo-Riemannian and hence it includes pseudo-Riemannian.

[49] The quadratic nature is represented by $dx^i dx^j$ and that is why the square $(ds)^2$ of the line element ds of Riemannian space is called the quadratic form.

use Lorentz transformations of spacetime coordinates to go from one inertial frame to another inertial frame (i.e. find the spacetime coordinates of the second frame from the spacetime coordinates of the first frame).

8. The covariant basis vectors of a given space are transformed from an unprimed coordinate system O to a primed coordinate system O' by the relation $\mathbf{E}'_i = \frac{\partial x^j}{\partial x'^i}\mathbf{E}_j$. Justify this relation and do the same for contravariant basis vectors.

Answer: The infinitesimal position differential (or infinitesimal displacement) is an invariant geometric property of the space (see the upcoming note 1) and hence it is independent of the coordinate system. Hence, we have:

$$\begin{aligned}
d\mathbf{r}' &= d\mathbf{r} \\
\frac{\partial \mathbf{r}'}{\partial x'^i}dx'^i &= \frac{\partial \mathbf{r}}{\partial x^j}dx^j \\
dx'^i\mathbf{E}'_i &= dx^j\mathbf{E}_j \\
dx'^i\mathbf{E}'_i &= \frac{\partial x^j}{\partial x'^i}dx'^i\mathbf{E}_j \\
\mathbf{E}'_i &= \frac{\partial x^j}{\partial x'^i}\mathbf{E}_j
\end{aligned}$$

where $d\mathbf{r}'$ and $d\mathbf{r}$ represent the infinitesimal position differential in O' and O respectively, in line 2 we use the chain rule, in line 3 we use the definition of the covariant basis vector, and in line 4 we use the chain rule. In fact, this can be shown briefly as:

$$\mathbf{E}'_i = \frac{\partial \mathbf{r}'}{\partial x'^i} = \frac{\partial \mathbf{r}}{\partial x'^i} = \frac{\partial \mathbf{r}}{\partial x^j}\frac{\partial x^j}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i}\mathbf{E}_j$$

For contravariant basis vectors we have:[50]

$$\begin{aligned}
d\mathbf{r}' &= d\mathbf{r} \\
dx'_i\mathbf{E}'^i &= dx_j\mathbf{E}^j \\
dx'_i\mathbf{E}'^i &= \frac{\partial x_j}{\partial x'_i}dx'_i\mathbf{E}^j \\
\mathbf{E}'^i &= \frac{\partial x_j}{\partial x'_i}\mathbf{E}^j \\
\mathbf{E}'^i &= \frac{\partial x'^i}{\partial x^j}\mathbf{E}^j
\end{aligned}$$

where in the last step we use the fact that a subscript/superscript in the numerator/denominator is equivalent to a superscript/subscript in the denominator/numerator (and vice versa)[51] while the other steps are justified as in the covariant case. In fact, this can be shown briefly as:

$$\mathbf{E}'^i = \nabla x'^i = \frac{\partial x'^i}{\partial x_k} = \frac{\partial x'^i}{\partial x^j}\frac{\partial x^j}{\partial x_k} = \frac{\partial x'^i}{\partial x^j}\nabla x^j = \frac{\partial x'^i}{\partial x^j}\mathbf{E}^j$$

where x_k represents Cartesian coordinates.

Note 1: the invariance of the infinitesimal position differential is closely related to the invariance of the line element where both represent a geometric property of the space that is independent of any coordinate system or frame of reference. Hence, $d\mathbf{r}'$ and $d\mathbf{r}$ may be replaced with $d\mathbf{s}'$ and $d\mathbf{s}$ where the latter symbols are defined as the vector form of the line element in O' and O. In brief, ds is the magnitude of the vector $d\mathbf{r}$ as can be easily inferred from the relation $(ds)^2 = d\mathbf{r}\cdot d\mathbf{r}$.

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[50] Subscripted x and x' are used here to refer to general coordinates in their covariant form.

[51] We note that this applies to general coordinates although it also applies, but trivially, to Cartesian coordinates.

Note 2: the relation $d\mathbf{r} = dx_j \mathbf{E}^j$ (and similarly $d\mathbf{r}' = dx'_i \mathbf{E}'^i$) is justified by the chain rule, i.e. $d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial x_j} dx_j = dx_j \mathbf{E}^j$ where we use the definition of the contravariant basis vector (since $\frac{\partial \mathbf{r}}{\partial x_j} \equiv \nabla x^j$).
Note 3: since labeling the systems as primed and unprimed is arbitrary, the transformation rules from a primed coordinate system O' to an unprimed coordinate system O can be obtained from the above relations by exchanging the primed and unprimed symbols, that is:

$$\begin{aligned}\mathbf{E}_i &= \frac{\partial x'^j}{\partial x^i} \mathbf{E}'_j \\ \mathbf{E}^i &= \frac{\partial x^i}{\partial x'^j} \mathbf{E}'^j\end{aligned}$$

9. Which space can be coordinated by rectilinear coordinate systems and which space cannot?
Answer: Flat space can be coordinated by rectilinear coordinate system (as well as by curvilinear coordinate system) while curved space cannot be coordinated by rectilinear coordinate system (and hence it must be coordinated by curvilinear coordinate system). In brief, all spaces can be coordinated by curvilinear coordinate systems but only flat spaces can be coordinated by rectilinear coordinate systems.
10. Show that $\mathbf{E}_j \cdot \mathbf{E}^i = \mathbf{E}^i \cdot \mathbf{E}_j = \delta^i_j$ (where $\mathbf{E}^i$ and $\mathbf{E}_j$ are the basis vectors and δ^i_j is the Kronecker delta).
Answer: We have:

$$\begin{aligned}\mathbf{E}_j \cdot \mathbf{E}^i &= \mathbf{E}^i \cdot \mathbf{E}_j \\ &= \nabla x^i \cdot \frac{\partial \mathbf{r}}{\partial x^j} \\ &= \left(\frac{\partial x^i}{\partial x_1}, \cdots, \frac{\partial x^i}{\partial x_n}\right) \cdot \left(\frac{\partial x_1}{\partial x^j}, \cdots, \frac{\partial x_n}{\partial x^j}\right) \\ &= \frac{\partial x^i}{\partial x_1}\frac{\partial x_1}{\partial x^j} + \cdots + \frac{\partial x^i}{\partial x_n}\frac{\partial x_n}{\partial x^j} \qquad \text{(no sum on } n) \\ &= \frac{\partial x^i}{\partial x_k}\frac{\partial x_k}{\partial x^j} \\ &= \frac{\partial x^i}{\partial x^j} \\ &= \delta^i_j\end{aligned}$$

where line 1 is justified by the commutativity of the inner product of vectors, in line 2 we use the definitions of $\mathbf{E}^i$ and $\mathbf{E}_j$, in line 3 we use the definitions of ∇ and $\mathbf{r}$ (in Cartesian coordinates which are represented by the subscripted symbols x_1, x_n, x_k), in line 4 we use the definition of inner product of vectors (i.e. sum of products of corresponding components), in line 5 we use the summation convention (with $k = 1, \cdots, n$), in line 6 we use the chain rule of differentiation, and in line 7 we use a well known tensor identity (which is based on the mutual independence of coordinates, i.e. $\frac{\partial x^i}{\partial x^j} = 1$ when $i = j$ and $\frac{\partial x^i}{\partial x^j} = 0$ when $i \neq j$) noting that an upper index in the denominator is equivalent to a lower index in the numerator (where this applies to general coordinates although it also applies, but trivially, to Cartesian coordinates).
11. Why should we invent (and hence use) different types of coordinate systems which leads for example to the complications of transformation and to potential confusion?
Answer: Each coordinate system (including frames of reference which are the coordinate systems of spacetime) have certain features and hence in any given situation it can be advantageous (or even necessary) to use a particular type of coordinate system in a given space or physical situation rather than another type. So, the choice and use of a particular system is determined by the circumstances and contexts. For example:
• A particular type of coordinate system may be mathematically simpler, e.g. Cartesian systems are mathematically simpler and hence they are generally easier to use than other types of coordinate

system. So, if the Cartesian system is sufficient to describe the geometric or physical situation then it is beneficial to use it instead of using more complicated systems which require more effort and can lead to confusion and error.

- A particular type of coordinate system can match (and hence demonstrate) certain geometric and physical features of the problem in hand. For example, if we have a geometric or physical problem with spherical/cylindrical symmetry then it is beneficial to use spherical/cylindrical coordinate system to match the symmetry of the problem and hence simplify the solution and reduce the effort, confusion and probability of error (as well as demonstrating this important geometric feature).
- A particular type of coordinate system can behave badly at certain points or regions in the space and hence it can be problematic in describing the problem in hand. For example, some coordinate systems have accidental singularities at certain points and hence if the domain of our problem includes those points of singularity then obtaining a solution can become difficult or impossible and hence we should use another type of coordinate system that does not have singularities in our domain. In fact, this is demonstrated (as we will see later) in general relativity by the Schwarzschild coordinate system which has an accidental (or coordinate) singularity at the Schwarzschild radius (i.e. $\frac{2GM}{c^2}$) and this singularity can be removed by using other types of coordinate system (e.g. Kruskal-Szekeres coordinates).

Note: as indicated above, choosing a certain type of coordinate systems may sometime be a matter of necessity and not a matter of preference. For example, when the space is curved we cannot use Cartesian coordinate systems (i.e. globally) since they cannot reflect and represent the curved nature of the geometry of space.

12. Discuss the relation between the invariance principle and coordinate transformations and link this to the nature of the coordinate systems and physical laws.

Answer: The relation between the invariance principle and coordinate transformations is highly dependent on the nature of the coordinate systems and the dynamic relation between them. To be clear, let investigate the following cases:

(**a**) Two coordinate systems that are static (i.e. at rest relative to each other). It should be obvious that any (alleged) physical law that is not dependent on position (i.e. difference in coordinates) should be invariant across these coordinate systems because there is nothing in the law that can discriminate between these systems. However, any alleged law that depends on position should not be invariant and hence it should be rejected by the invariance principle.

(**b**) Two coordinate systems that are in relative uniform motion. It should be obvious that any (alleged) physical law that is not dependent on position or velocity (i.e. first order time derivative of position) should be invariant across these coordinate systems because there is nothing in the law that can discriminate between these systems. However, the existence of a valid transformation between these systems that transforms such a law invariantly is not self-evident and hence we need to propose a specific transformation and show that the law actually transforms invariantly across these systems by the proposed transformation. In fact, this is the case with Lorentz mechanics where inertial frames of relative uniform motion represent coordinate systems of the spacetime and where Lorentz transformations were proposed and demonstrated to transform the laws of mechanics and electromagnetism (at least) invariantly. It should be obvious that Lorentz mechanics could not have been constructed by just accepting the principle of invariance and claiming that there should be a transformation that can transform these laws invariantly (possibly because we can put our laws in certain form, e.g. tensor form).

(**c**) Two coordinate systems that are in relative non-uniform motion. It should be obvious that the invariance and non-invariance of any (alleged) physical law across these systems depends on the order of the time derivative(s) of position which the law depends on and on the nature of the non-uniform motion. For example, if the law depends on the third order time derivative of position while the type of the non-uniform motion is represented by the second order time derivative then the law should be invariant, but if the law depends on the second order time derivative of position while the type of the non-uniform motion is represented by the second and third order time derivatives then the law should not be invariant. However, as before, the existence of a valid transformation between these systems that transforms such a law invariantly is not self-evident and hence we need to propose a specific transformation and show that the law actually transforms invariantly across these systems by the proposed transformation. In fact, this is the case with general relativity where it should propose and demonstrate that such transformations do exist and they can transform such laws invariantly across such coordinate systems (or frames). So, it is not enough to "implement" the invariance principle by dictating the condition that the laws should be cast in a certain form because such "implementation" is (at least) void and useless. In other words, imposing the requirement of being cast in tensor form is not sufficient to satisfy the demand of invariance.

2.2 Tensors

1. Justify the fact that the number of components of a rank-r tensor in an nD space is given by n^r.
Answer: Being a rank-r tensor means that the number of its free indices is r where each one of these indices ranges over $1, \cdots, n$ since it is in an nD space. Accordingly, the number of various possibilities of the indices (where each one of these possibilities represents a component) is given by the permutations (including the repetitive ones) of the values of indices, that is:

$$n \times n \times \cdots \times n = n^r$$

since we have r factors on the left hand side.

2. Justify the fact that the number of independent components of rank-2 symmetric tensors is $\frac{n(n+1)}{2}$ while the number of independent non-zero[52] components of rank-2 anti-symmetric tensors is $\frac{n(n-1)}{2}$.
Answer: A rank-2 tensor in an nD space has n^2 components because such a tensor is like an $n \times n$ matrix and hence it should have $n \times n = n^2$ components. In other words, we have n possibilities for the first index and n possibilities for the second index and hence we have $n \times n$ possibilities for the components of the tensor. This is also justified more generally by the answer of the previous exercise. Now, if the tensor is symmetric then we have n diagonal independent components plus half of the remaining $\left(n^2 - n\right)$ components (where they are halved because of the correlation $A_{ij} = A_{ji}$) and hence the total number of independent components is:

$$n + \frac{n^2 - n}{2} = \frac{2n + n^2 - n}{2} = \frac{n^2 + n}{2} = \frac{n\,(n+1)}{2}$$

On the other hand, if the tensor is anti-symmetric then the diagonal elements are identically zero[53] and only half of the remaining $\left(n^2 - n\right)$ components are independent (due to the correlation $A_{ij} = -A_{ji}$). Accordingly, the number of independent non-zero components of rank-2 anti-symmetric tensors is:

$$\frac{n^2 - n}{2} = \frac{n\,(n-1)}{2}$$

3. Find the transformation rules of a vector $\mathbf{A}$ between unprimed and primed coordinate systems and hence generalize these rules to tensors of any rank.
Answer: For contravariant vector we have:

$$\begin{aligned} \mathbf{A}' &= \mathbf{A} \\ \mathbf{E}'_j A'^j &= \mathbf{E}_i A^i \\ \mathbf{E}'_j A'^j &= \frac{\partial x'^j}{\partial x^i} \mathbf{E}'_j A^i \\ A'^j &= \frac{\partial x'^j}{\partial x^i} A^i \end{aligned}$$

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[52] "Non-zero" here means they are not necessarily zero due to the anti-symmetry requirement although they could be zero accidentally or for another reason.

[53] This is because they should be equal to their negative (i.e. $A_{ii} = -A_{ii}$) according to the requirement of anti-symmetry and hence they must be zero.

where in line 1 we use the fact that **A** is a vector (i.e. rank-1 tensor) and hence geometrically it is the same in both systems, and in line 3 we use a result from exercise 8 of § 2.1. By exchanging the primes we similarly get $A^j = \frac{\partial x^j}{\partial x'^i} A'^i$.

For covariant vector we have:

$$\begin{aligned} \mathbf{A}' &= \mathbf{A} \\ \mathbf{E}'^j A'_j &= \mathbf{E}^i A_i \\ \mathbf{E}'^j A'_j &= \frac{\partial x^i}{\partial x'^j} \mathbf{E}'^j A_i \\ A'_j &= \frac{\partial x^i}{\partial x'^j} A_i \end{aligned} \tag{5}$$

where in line 3 we use a result from exercise 8 of § 2.1. By exchanging the primes we similarly get $A_j = \frac{\partial x'^i}{\partial x^j} A'_i$.

The generalization of the above transformation rules is straightforward using the same method as above. For example, the rule for transforming a rank-2 mixed tensor **B** from O to O' can be found as follows:

$$\begin{aligned} \mathbf{B}' &= \mathbf{B} \\ \mathbf{E}'_m \mathbf{E}'^n B'^m{}_n &= \mathbf{E}_i \mathbf{E}^j B^i{}_j \\ \mathbf{E}'_m \mathbf{E}'^n B'^m{}_n &= \frac{\partial x'^m}{\partial x^i} \mathbf{E}'_m \frac{\partial x^j}{\partial x'^n} \mathbf{E}'^n B^i{}_j \\ B'^m{}_n &= \frac{\partial x'^m}{\partial x^i} \frac{\partial x^j}{\partial x'^n} B^i{}_j \end{aligned}$$

where in line 3 we use the transformation rules of the basis vectors (see exercise 8 of § 2.1). Accordingly, we conclude that the transformation rules of tensors of any rank are as given by Eq. 41 in the text (noting that this includes even rank-0 tensors where we have $A' = A$).

4. Distinguish between covariant and contravariant tensors using their transformation rules.

Answer: For simplicity and clarity, we use vectors (i.e. rank-1 tensors) as an example to demonstrate the distinction between covariant and contravariant tensors (the generalization to higher rank tensors should be obvious). Covariant and contravariant vectors are transformed according to the following rules respectively:

$$A'_i = \frac{\partial x^j}{\partial x'^i} A_j \qquad \text{and} \qquad A'^i = \frac{\partial x'^i}{\partial x^j} A^j \tag{6}$$

These rules can be written as:[54]

$$A'_i \partial x'^i = A_j \partial x^j \qquad \text{and} \qquad A'^i \partial x^j = A^j \partial x'^i$$

As we see, in the covariant case the primed symbols are on one side and the unprimed symbols are on the other side, while in the contravariant case the primed and unprimed symbols are on both sides. This indicates an important difference between the two cases, that is the proportionality between the tensor and coordinate differential is inverse in one case and direct in the other case. In fact, all the real and conventional differences between covariance and contravariance are based on this fundamental difference in the nature of this proportionality relation where this nature originates from the type of the basis vectors (i.e. tangent or gradient) to which the components of the tensor are referred.

Note 1: we may also write the above relations as:

$$A'_i \partial x'^i = A_j \partial x^j \qquad \text{and} \qquad \frac{A'^i}{\partial x'^i} = \frac{A^j}{\partial x^j}$$

---

[54] The following relations should not be seen as rigorous mathematical equations but rather as demonstration to the qualitative difference in the nature of covariant and contravariant tensors.

where we keep the primed and unprimed symbols on different sides in both cases. This similarly demonstrates the aforementioned difference in the nature of the proportionality relation. In fact, we may even write these relations as (noting the difference between the covariant and contravariant coordinates):

$$A'_i \partial x'^i = A_j \partial x^j \qquad \text{and} \qquad A'^i \partial x'_i = A^j \partial x_j$$

We may also write the above relations as:

$$\frac{A'_i}{A_j} = \frac{\partial x^j}{\partial x'^i} \qquad \text{and} \qquad \frac{A'^i}{A^j} = \frac{\partial x'^i}{\partial x^j}$$

The last relation may be the most obvious in demonstrating the aforementioned proportionality relationship.

Note 2: we may also distinguish between covariant and contravariant tensors using their transformation rules but in a different way. If we note that $\frac{\partial x^j}{\partial x'^i}$ and $\frac{\partial x'^i}{\partial x^j}$ represent inverse or reciprocal transformations then from Eq. 6 we can conclude that covariant and contravariant vectors (and hence tensors) use reciprocal rules in their transformation from one system to another, i.e. one type uses a direct rule of transformation while the other type uses a reciprocal rule of transformation.

5. Show that the zero tensor is value-invariant (i.e. constant) across all coordinate systems.

Answer: Let have an arbitrary tensor $A^{i\cdots m}_{j\cdots n}$ which is zero in the unprimed coordinate system O and we want to transform it to a primed coordinate system O'. Accordingly, we have:

$$A'^{I\cdots M}_{J\cdots N} = \frac{\partial x'^I}{\partial x^i} \cdots \frac{\partial x'^M}{\partial x^m} \frac{\partial x^j}{\partial x'^J} \cdots \frac{\partial x^n}{\partial x'^N} A^{i\cdots m}_{j\cdots n}$$

Now, since $A^{i\cdots m}_{j\cdots n} = 0$ then $A'^{I\cdots M}_{J\cdots N} = 0$ (i.e. the tensor is also zero in the primed coordinate system O'). Because the system O' is arbitrary the result should apply to all coordinate systems.

6. What you notice about the tensor relation $A^{i\cdots m}_{j\cdots n} = 0$?

Answer: First, it obviously means that $A^{i\cdots m}_{j\cdots n}$ is a zero tensor. Like other similar tensor relations, this relation can be seen as a component equation (i.e. the components of the tensor **A** are zero) and may also be seen as a tensor equation (i.e. equivalent to $\mathbf{A} = \mathbf{0}$) where the latter is justified by the common convention that the zero tensor (and indeed any constant tensor whose all components are the same) does not require indices, i.e. we do not need to write $A^{i\cdots m}_{j\cdots n} = 0^{i\cdots m}_{j\cdots n}$ where $0^{i\cdots m}_{j\cdots n}$ is the zero tensor (and similarly we do not need to write $B^{i\cdots m}_{j\cdots n} = c^{i\cdots m}_{j\cdots n}$ where $c^{i\cdots m}_{j\cdots n}$ is a constant tensor whose all components are c).

Note: "constant tensor" may be used in the literature in two different meanings:

(**a**) The individual components of the tensor are constants although these components are different from each other. For example, $\eta_{\mu\nu} = \text{diag}\,[1, -1, -1, -1]$ is constant in this sense.

(**b**) The tensor itself and in its entirety is constant, i.e. all its components have the same constant value. For example, the zero tensor $A^{i\cdots m}_{j\cdots n} = 0$ is constant in this sense.

So, the reader of the literature should be aware of this to avoid potential confusion.

7. Show that linear combinations of tensors are tensors.

Answer: For example, if in O system $C^i{}_j$ and $D^i{}_j$ are rank-2 mixed tensors and their linear combination is $aC^i{}_j \pm bD^i{}_j$ (where a and b are scalars) then in O' system we have:

$$C'^k{}_l = \frac{\partial x'^k}{\partial x^i} \frac{\partial x^j}{\partial x'^l} C^i{}_j \qquad \text{and} \qquad D'^k{}_l = \frac{\partial x'^k}{\partial x^i} \frac{\partial x^j}{\partial x'^l} D^i{}_j$$

Hence:

$$aC'^k{}_l \pm bD'^k{}_l = a\frac{\partial x'^k}{\partial x^i} \frac{\partial x^j}{\partial x'^l} C^i{}_j \pm b\frac{\partial x'^k}{\partial x^i} \frac{\partial x^j}{\partial x'^l} D^i{}_j = \frac{\partial x'^k}{\partial x^i} \frac{\partial x^j}{\partial x'^l} \left(aC^i{}_j \pm bD^i{}_j\right)$$

As we see, the linear combination $aC^i{}_j \pm bD^i{}_j$ transforms as a tensor and hence it is a tensor. The pattern can be easily generalized to any linear combination of tensors of any rank and of any variance type.

8. Show that outer products of tensors are tensors.
Answer: For example, if in O system C^i and D_j are rank-1 tensors and their outer product is $C^i D_j$ then in O' system we have:

$$C'^k = \frac{\partial x'^k}{\partial x^i} C^i \qquad \text{and} \qquad D'_l = \frac{\partial x^j}{\partial x'^l} D_j$$

Therefore:

$$C'^k D'_l = \frac{\partial x'^k}{\partial x^i} C^i \frac{\partial x^j}{\partial x'^l} D_j = \frac{\partial x'^k}{\partial x^i} \frac{\partial x^j}{\partial x'^l} \left(C^i D_j \right)$$

As we see, the outer product $C^i D_j$ transforms as a tensor and hence it is a tensor. The pattern can be easily generalized to outer products of tensors of any rank and of any variance type.

9. Show that contractions of tensors (i.e. mixed tensors of rank > 1) are tensors.
Answer: We use a rank-3 mixed tensor $A^i{}_{jk}$ as an example to get the pattern and generalize. According to the transformation rules of tensors, $A^i{}_{jk}$ is transformed as:

$$A'^a{}_{bc} = \frac{\partial x'^a}{\partial x^i} \frac{\partial x^j}{\partial x'^b} \frac{\partial x^k}{\partial x'^c} A^i{}_{jk}$$

On contracting a and b we get:

$$A'^a{}_{ac} = \frac{\partial x'^a}{\partial x^i} \frac{\partial x^j}{\partial x'^a} \frac{\partial x^k}{\partial x'^c} A^i{}_{jk} = \frac{\partial x^j}{\partial x^i} \frac{\partial x^k}{\partial x'^c} A^i{}_{jk} = \delta^j_i \frac{\partial x^k}{\partial x'^c} A^i{}_{jk} = \frac{\partial x^k}{\partial x'^c} A^i{}_{ik}$$

where in equality 2 we use the chain rule while in equality 4 we use δ^j_i for index replacement. As we see, the last equality represents the transformation rule of a rank-1 tensor from unprimed system to primed system. The pattern can be easily captured and generalized to conclude that contractions of tensors are tensors.

10. Show that inner products of tensors are tensors.
Answer: Inner product can be regarded as an outer product followed by contraction, e.g.

$$A_{ij} B^{jn} = \delta^j_m \left(A_{ij} B^{mn} \right)$$

Now, according to the last two exercises both these operations preserve the tensorial nature and hence the inner product of tensors should be a tensor.

11. Show that permutations of tensors are tensors.
Answer: Let have a rank-2 covariant tensor A_{ij} in system O and hence it is transformed to system O' by the rule:

$$A'_{ab} = \frac{\partial x^i}{\partial x'^a} \frac{\partial x^j}{\partial x'^b} A_{ij}$$

On permuting the indices we get:

$$A'_{ba} = \frac{\partial x^j}{\partial x'^b} \frac{\partial x^i}{\partial x'^a} A_{ji}$$

which is the transformation rule of a tensor A_{ji} from system O to system O'.[55] This should similarly apply to permutation operations in tensors of other ranks and variance types.

12. Show that the symmetry properties (i.e. being symmetric/anti-symmetric/neither) of tensors are invariant under coordinate transformations.
Answer: Let A_{ij} be a symmetric tensor in system O and hence $A_{ij} = A_{ji}$. If we transform A_{ij} to system O' then we have:

$$A'_{ab} = \frac{\partial x^i}{\partial x'^a} \frac{\partial x^j}{\partial x'^b} A_{ij} = \frac{\partial x^i}{\partial x'^a} \frac{\partial x^j}{\partial x'^b} A_{ji} = \frac{\partial x^j}{\partial x'^b} \frac{\partial x^i}{\partial x'^a} A_{ji} = A'_{ba}$$

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[55] In fact, this is just a demonstration that the permutation (as viewed from the relation between the two systems) is no more than a trivial relabeling operation.

where the second equality is because $A_{ij} = A_{ji}$ while in the last equality we use the transformation rule of A_{ji} from O to O' since A_{ji} is also a tensor (as shown in the previous exercise). Accordingly, A'_{ab} is also symmetric.

If A_{ij} is anti-symmetric then $A_{ij} = -A_{ji}$ and hence we have:

$$A'_{ab} = \frac{\partial x^i}{\partial x'^a}\frac{\partial x^j}{\partial x'^b}A_{ij} = -\frac{\partial x^i}{\partial x'^a}\frac{\partial x^j}{\partial x'^b}A_{ji} = -\frac{\partial x^j}{\partial x'^b}\frac{\partial x^i}{\partial x'^a}A_{ji} = -A'_{ba}$$

It should be obvious that the above pattern (in both the symmetric and anti-symmetric cases) applies regardless of the rank and variance type of the tensor.

Regarding the invariance property of being "neither" (i.e. neither symmetric nor anti-symmetric), it can be easily inferred from the invariance property of being symmetric/anti-symmetric because if a tensor is "neither" in system O and it is transformed to a symmetric/anti-symmetric tensor in system O' then by the invariance property of being symmetric/anti-symmetric (as applied in system O') it should also be symmetric/anti-symmetric in system O which is a contradiction.

Hence, we can conclude that the symmetry properties of tensors are invariant under coordinate transformations, as required.

13. Why in the contraction operation of tensors the contracted indices should be of opposite variance type (i.e. one upper and one lower)?

    Answer: This is to keep the invariance property of tensors. Let demonstrate this by an example from vectors which should be sufficient to clarify the point (the generalization is straightforward). If **A** is a vector transformed from system O to system O' then we have:

    $$A'^i A'_i = \frac{\partial x'^i}{\partial x^j}A^j \frac{\partial x^k}{\partial x'^i}A_k = \frac{\partial x'^i}{\partial x^j}\frac{\partial x^k}{\partial x'^i}A^j A_k = \frac{\partial x^k}{\partial x^j}A^j A_k = \delta^k_j A^j A_k = A^k A_k$$

    i.e. the contracted tensor is the same in the two systems. In contrast:

    $$\begin{aligned} A'^i A'^i &= \frac{\partial x'^i}{\partial x^j}A^j \frac{\partial x'^i}{\partial x^k}A^k = \frac{\partial x'^i}{\partial x^j}\frac{\partial x'^i}{\partial x^k}A^j A^k \\ A'_i A'_i &= \frac{\partial x^j}{\partial x'^i}A_j \frac{\partial x^k}{\partial x'^i}A_k = \frac{\partial x^j}{\partial x'^i}\frac{\partial x^k}{\partial x'^i}A_j A_k \end{aligned}$$

    i.e. the results are not invariant since the transformations depend on the systems involved.

14. Show that all the variance types of a tensor are tensors.

    Answer: Let A'^{mn} be a tensor in system O' transformed from system O. On lowering the index m we have:

    $$\begin{aligned} A'{}_m{}^n &= g'_{ma}A'^{an} \\ &= g_{ib}\frac{\partial x^i}{\partial x'^m}\frac{\partial x^b}{\partial x'^a}A'^{an} \\ &= g_{ib}\frac{\partial x^i}{\partial x'^m}\frac{\partial x^b}{\partial x'^a}\frac{\partial x'^a}{\partial x^c}\frac{\partial x'^n}{\partial x^j}A^{cj} \\ &= g_{ib}\frac{\partial x^i}{\partial x'^m}\frac{\partial x^b}{\partial x^c}\frac{\partial x'^n}{\partial x^j}A^{cj} \\ &= g_{ib}\frac{\partial x^i}{\partial x'^m}\delta^b_c\frac{\partial x'^n}{\partial x^j}A^{cj} \\ &= g_{ib}\frac{\partial x^i}{\partial x'^m}\frac{\partial x'^n}{\partial x^j}A^{bj} \\ &= \frac{\partial x^i}{\partial x'^m}\frac{\partial x'^n}{\partial x^j}A_i{}^j \end{aligned}$$

    where in line 2 we transform the metric tensor since it is a rank-2 tensor (see exercise 4 of § 2.5), in line 3 we use the transformation rule of A'^{an}, in line 4 we use the chain rule, in line 5 we use the fact

that the coordinates are mutually independent, in line 6 we use δ^b_c for index replacement operation, and in line 7 we use g_{ib} for index lowering operation. As we see, $A'^{\ n}_{m}$ also transforms from O to O' as a tensor and hence it is a tensor. The above method can be repeated on index raising operation and the pattern can be generalized to any non-scalar tensor regardless of its rank and variance type and regardless of the number of index shifting operations (since each operation preserves the tensorial nature). Hence, we conclude that all variance types of a (non-scalar) tensor are tensors, as required.
Note 1: a more simple proof is that all variance types of a tensor are obtained by index shifting operations which are inner product operations between the metric *tensor* (which is a tensor) and another tensor and hence the result must be a tensor according to exercise 10.
Note 2: it should be obvious that index replacement operation does not change the tensorial nature of the tensor whose index is replaced.[56] This may be justified formally by noting that index replacement operation is an inner product operation between the Kronecker delta *tensor* (which is a tensor; see exercise 5 of § 2.5) and the tensor whose index is replaced (e.g. $A_j = \delta^i_j A_i$) and hence the result must be a tensor according to exercise 10.

15. Discuss the nabla and Laplacian operators in 4D spacetime.
Answer:[57] The nabla 3-operator ∇, which is a 3D spatial vector operator, can be extended to the 4D spacetime by adding the temporal coordinate where it is symbolized as $\Box$ and is given in its covariant and contravariant forms by:

$$[\partial_\mu] = \left[\frac{\partial}{\partial x^\mu}\right] \equiv \left[\frac{\partial}{\partial x^0}, \nabla\right] \qquad \text{and} \qquad [\partial^\mu] = \left[\frac{\partial}{\partial x_\mu}\right] \equiv \left[\frac{\partial}{\partial x_0}, -\nabla\right]$$

where the nabla 3-operator ∇ represents the three spatial components and $\partial^\mu = g^{\mu\nu}\partial_\nu$ (noting that the above form corresponds to $g^{\mu\nu} = \eta^{\mu\nu}$). The mostly used 3D coordinate system is the rectangular Cartesian for which the nabla 3-operator is:

$$\nabla = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]$$

and hence the nabla 4-operator becomes $\left[\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]$ in its covariant form and $\left[\frac{1}{c}\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right]$ in its contravariant form.
Similarly, the Laplacian 3-operator ∇^2, which is a 3D spatial scalar operator, can be extended to the 4D spacetime by adding the temporal coordinate where it is symbolized as $\Box^2$ and is given as an inner product of the nabla 4-operator by itself, i.e. covariant by contravariant.[58] Accordingly, the Laplacian 4-operator (which is also known as d'Alembert operator or the d'Alembertian) is given by:

$$\Box^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

where ∇^2 is the ordinary spatial Laplacian 3-operator as defined in the mathematical textbooks for various coordinate systems. The mostly used 3D coordinate system is the rectangular Cartesian for which the 3D spatial Laplacian is:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

and hence the d'Alembertian becomes:

$$\Box^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \tag{7}$$

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[56] In fact, this is a trivial statement. So, it is added for pedagogical purpose (as well as potential clarity and completeness).

[57] This answer is mainly taken from B4 and hence it is mostly related to flat spacetime of special relativity. In fact, some notations and formulations here require generalization but we do not do that because what we need in this book from these operators is mainly related to linearized formulation corresponding to quasi-Minkowskian spacetime and hence we do not need to get involved in unnecessary details and complications.

[58] Product of differential operators means consecutive differentiation.

In a more general tensor notation where the underlying coordinate system could be curvilinear, the d'Alembertian is given by:

$$\Box^2 \equiv \partial^\mu \partial_\mu = g^{\mu\nu} \partial_\nu \partial_\mu = g_{\mu\nu} \partial^\nu \partial^\mu = \partial_\mu \partial^\mu$$

In fact, this tensor form also applies to the above definition of the d'Alembertian (i.e. Eq. 7) where $g^{\mu\nu} = \eta^{\mu\nu}$ and $g_{\mu\nu} = \eta_{\mu\nu}$.

2.3 Curvature of Space

1. Justify the indicators and conditions (for testing the intrinsic curvature and flatness of space) that we stated in the Problems.
   Answer: We may say:
   • Special geometric tests: these are based on our fundamental definitions and conventions about "flat space" and "curved space" as well as our direct experiences and conventions about fundamental geometric shapes (such as triangle and circle) and the system of measurements. Ultimately, these are based on our postulates which cannot be proved from more fundamental principles.
   • Coordination by Cartesian system (i.e. globally): this may also be based on fundamental definitions and conventions, as well as our direct experiences, as in the special geometric tests indicator.
   • Riemann-Christoffel curvature tensor: this can be proved formally (see problem 3 of § 2.10) where the proof is ultimately based on the coordination by Cartesian system indicator (as well as basic definitions and facts).
   • Parallel transport: this can be proved formally (see exercise 4 of § 2.8) where the proof is based on the path-dependency of parallel transport in curved spaces.
   • Metric tensor: this can be proved formally (see exercise 16 of § 2.5) where the proof is based on the coordination by the Cartesian system indicator.
   • Christoffel symbols: this can be proved formally (see exercise 8 of § 2.6) where the proof is based on the coordination by Cartesian system indicator (or the metric tensor indicator).
2. Give an example of an indicator for testing the intrinsic curvature and flatness of space other than those given in the Problems.
   Answer: An example is the commutativity of covariant differential operators (i.e. they are commutative in flat space and non-commutative in curved space) as will be seen in § 2.10.[59] However, this is related to the Riemann-Christoffel curvature tensor indicator.
3. Give an example of a special geometric test (other than those given in the Problems) for testing the intrinsic curvature and flatness of space.
   Answer: For example, in curved space at least some initially-parallel geodesic curves do not stay parallel while in flat space all initially-parallel geodesic curves do stay parallel.
   Note: initial parallelism should belong to the tangent space at the point of parallelism.
4. How do you define triangle and circle in curved spaces?
   Answer: A triangle is a closed curve made of 3 distinct geodesic curves (i.e. its sides) that meet mutually in 3 distinct points (i.e. vertices). The angles of the triangle are then defined as the angles between the tangent lines[60] to the sides at the vertices of the triangle.
   A circle is a smooth closed curve whose all points have equal distance from a given point (i.e. center) in the space where the distance is defined as the length of the geodesic curve (i.e. radius) that connects a point on the circle to the center. The diameter is then defined as twice the length of the radius while the circumference is defined as the length of the circle.
5. In special and general relativity theories, what we mean by the curvature of space and why?
   Answer: Since we are inhabitants of the spacetime (which is the space of special and general relativity), then the curvature of space means intrinsic curvature. Hence, the space of special relativity is intrinsically flat while the space of general relativity is intrinsically curved. This can be seen from

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[59] A similar indicator may be obtained from the geodesic deviation condition (see § 7.13).

[60] This requires local flatness and the possibility of constructing a local Cartesian system which is guaranteed in Riemannian spaces.

characterizing the spacetime of these theories by the Riemann-Christoffel curvature tensor[61] which is a measure for the intrinsic (not extrinsic) curvature.

6. Give simple examples of how a 2D inhabitant of a surface can detect the intrinsic curvature of his 2D space by conducting simple measurements.
   Answer: The 2D inhabitant can measure lengths and angles (see the upcoming note 1). So, from his measurement of the length of the diameter D and the circumference C of a circle he can conclude that his space is flat if $C = \pi D$ and curved if $C \neq \pi D$. Also, from his measurements of angles of a triangle he can conclude that his space is flat if the sum is equal to π and curved if it is not equal to π.
   Note 1: lengths can be obtained from the line element of the space (which is intrinsic due to its sole dependency on the metric, i.e. $ds = \sqrt{g_{ij}dx^i dx^j}$) while angles can be obtained from the formula $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$ (where both inner product and length are intrinsic since both solely depend on the metric noting that $\mathbf{a}\cdot\mathbf{b} = g_{ij}a^i b^j = g^{ij}a_i b_j$). Accordingly, the measurement of lengths and angles is purely intrinsic process.
   Note 2: in the above tests we are assuming that the 2D space is flat or curved everywhere. So, if we have a space that is partly flat (i.e. it is flat in some parts and curved in other parts) then we should assume that the test corresponds to the particular region of the space in which the test takes place with no global implication. Yes, because flat conventionally means globally flat then discovering the curvedness in any part of the space qualifies the space to be curved although the opposite is not true (i.e. discovering the flatness in a given part of the space does not qualify the space to be flat unless this applies globally).
7. Briefly discuss the coordination by a Cartesian system as a sign for the curvedness and flatness of space.
   Answer: We characterize flat space as a space that can be coordinated by a Cartesian coordinate system globally; otherwise the space is curved. Accordingly, a space is flat *iff* it can be coordinated globally by a Cartesian coordinate system.
   We note that the condition "can be coordinated globally by a Cartesian coordinate system" is equivalent to the condition that there is a valid permissible coordinate transformation between the coordinate system of the space and a Cartesian coordinate system (where "transformation" should include the identity transformation as a special case). This may be expressed by saying: a Riemannian metric g_{ij} of a particular coordinate system is a Euclidean metric if it can be transformed to the identity tensor δ_{ij} by a permissible coordinate transformation (although this requires a trivial generalization to include the case of flat non-Euclidean).
8. How can the Riemann-Christoffel curvature tensor be used for testing the curvedness and flatness of space?
   Answer: A space is intrinsically flat *iff* its Riemann-Christoffel curvature tensor R_{ijkl} vanishes identically over the entire space. Hence, $R_{ijkl} \equiv 0$ is a sufficient and necessary condition for space flatness (and $R_{ijkl} \neq 0$ is a sufficient and necessary condition for space curvedness).
9. Flat spaces may be metricized by non-constant metric tensor (i.e. tensor whose components are functions of coordinates). For example, 3D Euclidean space can be coordinated by a spherical coordinate system whose metric g_{ij} is diag $\left[1, r^2, r^2\sin^2\theta\right]$. Comment on this.
   Answer: A flat space is characterized by being a space that "can" be metricized by a constant metric tensor. This does not mean that flat space "must" be metricized by a constant metric tensor (see the upcoming note). So, as long as the space can be metricized by a constant metric tensor (although it is not actually metricized by a constant metric tensor) it is flat. The key to this issue is that when the space is flat there should be a permissible coordinate transformation from any of its coordinate systems (whose metric is not constant) and a Cartesian system (whose metric is constant) and the latter characterizes space flatness.
   Note: the metric tensor does not depend on the geometry of the space only but it also depends on the employed coordinate system (see § 2.5). So, if the coordinate system is Cartesian (or rectilinear

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[61] See for example § 2.13 where the Einstein tensor is defined in terms of the Ricci curvature tensor and scalar which are derived from the Riemann-Christoffel curvature tensor.

to be more general) the metric tensor is constant; otherwise it is variable even if the space is flat. In other words, the metric tensor in rectilinear systems is constant while the metric tensor in curvilinear systems is variable (noting that rectilinear systems can coordinate only flat spaces while curvilinear systems can coordinate both flat spaces and curved spaces).

2.4 Variational Principle and Euler-Lagrange Equation

1. What is the relation between the variational principle and the Euler-Lagrange equation?
   Answer: The Euler-Lagrange equation is a relation that casts (or expresses) the variational principle in a specific mathematical form when the optimized function is a functional.
2. What is the role of the Euler-Lagrange equation in implementing the variational principle?
   Answer: If F is a functional of a function f [i.e. $F(f)$] and we want to extremize F by searching for a mathematical condition that makes f extremize F, then the role of the Euler-Lagrange equation is to find this mathematical condition. In brief, the role of the Euler-Lagrange equation is to find the mathematical condition that optimizes $F(f)$.

2.5 Metric Tensor

1. Explain the relation between space, coordinate system and metric tensor.
   Answer: We note the following:
   • The function of the coordinate system is to identify the points of the space uniquely and unambiguously, while the function of the metric tensor is to describe the geometry of the space by exploiting the coordinate system.
   • The coordinate system uniquely identifies its space but the space does not uniquely identify its coordinate system since the space can be coordinated by different coordinate systems.
   • Similarly, the metric tensor uniquely identifies its space but the space does not uniquely identify its metric tensor unless the space is associated with a specific coordinate system.
   • Accordingly, if we have a given space then we should first find a suitable coordinate system that identifies this space and from this coordinate system we derive the metric tensor that describes the geometry of the space. So, we can say: the coordinate system is a tool for obtaining the metric and hence describing the geometry of the space by its metric.
   • The above points can be clarified further by noting the relations: $g_{ij} = \mathbf{E}_i \cdot \mathbf{E}_j$ and $g^{ij} = \mathbf{E}^i \cdot \mathbf{E}^j$ where the covariant metric tensor is obtained from the inner product of the covariant basis vectors of the coordinate system, while the contravariant metric tensor is obtained from the inner product of the contravariant basis vectors of the coordinate system.[62] These relations demonstrate in a formal way the dependence of the metric tensor of the space on the employed coordinate system of the space (as represented by its basis vectors), although the essence of the metric tensor (as a representative of the space geometry) does not depend on any particular coordinate system. This should elucidate our statement: "the metric has geometric significance regardless of any coordinate system although it requires a coordinate system to be represented in a specific form". In fact, this is just a demonstration of the tensorial nature of the metric tensor (see exercise 4).
2. What are the physical dimensions of the metric tensor?
   Answer: Being an inner product of the basis vectors (as seen in the previous exercise) means that any component of the metric tensor can be expressed as a sum of products of partial derivatives of coordinates in one system with respect to coordinates in another system (i.e. $g_{ij} = \frac{\partial x_k}{\partial x^i} \frac{\partial x_k}{\partial x^j}$ and $g^{ij} = \frac{\partial x^i}{\partial x_k} \frac{\partial x^j}{\partial x_k}$ as seen earlier; also refer to the definition of basis vectors in § 2.1) and hence it is a ratio of coordinates to coordinates. Therefore, the metric tensor is dimensionless.
   Note: we should remark that the above explanation is rather simplistic since it is based on the assumption that all the coordinates of the system which the metric tensor is based upon have the physical dimensions of length (like the corresponding Cartesian system) which is not always the case

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[62] Similarly, the relation $g^i_j = \mathbf{E}_j \cdot \mathbf{E}^i = \mathbf{E}^i \cdot \mathbf{E}_j$ shows that the mixed metric tensor is obtained from the inner product between covariant and contravariant basis vectors.

since the coordinates of any system could have the same dimension (e.g. Cartesian whose all coordinates have the dimension of length) or different dimensions (e.g. spherical where some coordinates have the dimension of length while others are dimensionless). The important thing is that the line element (which is based on the metric) has always the dimension of length (or the quadratic form has always the dimension of length squared) regardless of the coordinates and their physical dimensions. Anyway, the results that we will obtain later about the physical dimensions of the quantities that are based on the metric tensor and coordinates will not change because we can assume that the obtained expressions for those quantities are based on a dimensionless metric tensor that is based on a coordinate system whose all coordinates have the same physical dimensions (i.e. length).
In fact, there are different approaches in the literature about distributing the physical dimensions in the definition of the quadratic form (or line element) between the metric and coordinates where some make the metric tensor dimensionless, others make the coordinates dimensionless, while some distribute the dimensions conveniently and hence the components of the metric (as well as the coordinates) have different physical dimensions.[63] However, all these approaches should be correct as long as they produce the correct dimensions for the physical quantities that are based on the metric and coordinates.[64]

3. Show that $g_{ij} = \mathbf{E}_i \cdot \mathbf{E}_j$.
Answer: According to the definition of the quadratic form we have $(ds)^2 = g_{ij} dx^i dx^j$. Also, from first principles we have:

$$(ds)^2 = d\mathbf{r} \cdot d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial x^i} \cdot \frac{\partial \mathbf{r}}{\partial x^j} dx^i dx^j = \mathbf{E}_i \cdot \mathbf{E}_j dx^i dx^j$$

where we use the chain rule in the second equality and the definition of the covariant basis vectors in the third equality. On comparing the above two expressions of $(ds)^2$ we conclude that $g_{ij} = \mathbf{E}_i \cdot \mathbf{E}_j$. In fact, this is just a demonstration of the link between the metric tensor, the basis vectors and the quadratic form rather than a proof of a mathematical statement. Yes, we may obtain this relation from the definitions of the metric tensor and the basis vectors (as well as the definition of the inner product), i.e. if we define the metric tensor by $g_{ij} \equiv \frac{\partial x^k}{\partial x^i} \frac{\partial x^k}{\partial x^j}$ and define the basis vectors by $\mathbf{E}_i \equiv \frac{\partial \mathbf{r}}{\partial x^i}$ and $\mathbf{E}_j \equiv \frac{\partial \mathbf{r}}{\partial x^j}$ then $g_{ij} = \mathbf{E}_i \cdot \mathbf{E}_j$. We should finally note that the relation $g^{ij} = \mathbf{E}^i \cdot \mathbf{E}^j$ can be similarly obtained by these methods.

4. Show that the metric is a rank-2 tensor.
Answer: To be a rank-2 tensor it should transform like a rank-2 tensor. Now, if g'_{ij} is the transformation of g_{mn} from one system (i.e. unprimed) to another system (i.e. primed) then we have:

$$\begin{aligned} g'_{ij} &= \mathbf{E}'_i \cdot \mathbf{E}'_j \\ &= \left(\frac{\partial x^m}{\partial x'^i} \mathbf{E}_m \right) \cdot \left(\frac{\partial x^n}{\partial x'^j} \mathbf{E}_n \right) \\ &= \frac{\partial x^m}{\partial x'^i} \frac{\partial x^n}{\partial x'^j} \mathbf{E}_m \cdot \mathbf{E}_n \\ &= \frac{\partial x^m}{\partial x'^i} \frac{\partial x^n}{\partial x'^j} g_{mn} \end{aligned}$$

where lines 1 and 4 are from the relation between the metric tensor and the basis vectors (see the previous exercise), and line 2 is from the result of exercise 8 of § 2.1. As we see, the last line is the transformation rule of a rank-2 covariant tensor (i.e. the metric g_{mn}) from the unprimed system to the primed system (whose metric is g'_{ij}). Hence, the metric is a rank-2 tensor.
Note: the tensorial nature of the contravariant metric tensor can be similarly established. It can also

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[63] We will find examples of the last approach in the upcoming exercises of the present section about the metric tensor of 3D Euclidean space in cylindrical and spherical coordinates (as well as the metric tensor of 4D flat spacetime of special relativity that employs cylindrical and spherical spatial coordinates). We will also find other examples in the Schwarzschild and Kerr metrics (see § 4.1 and § 4.2).

[64] In fact, some of these unpleasant complications should disappear if we adopt normalized physical coordinates (see B3).

be established by the metric rules, i.e. all variance types of a tensor are tensors (see exercise 14 of § 2.2).

5. Show that the Kronecker delta is a rank-2 tensor.
Answer: To be a rank-2 tensor it should transform like a rank-2 tensor. Now, if ${\delta'}^i_{\ j}$ is the transformation of $\delta^m_{\ n}$ from one system (i.e. unprimed) to another system (i.e. primed) then we have:

$$\begin{aligned}{\delta'}^i_{\ j} &= \mathbf{E}'^i \cdot \mathbf{E}'_j \\ &= \left(\frac{\partial x'^i}{\partial x^m}\mathbf{E}^m\right)\cdot\left(\frac{\partial x^n}{\partial x'^j}\mathbf{E}_n\right) \\ &= \frac{\partial x'^i}{\partial x^m}\frac{\partial x^n}{\partial x'^j}\mathbf{E}^m\cdot\mathbf{E}_n \\ &= \frac{\partial x'^i}{\partial x^m}\frac{\partial x^n}{\partial x'^j}\delta^m_{\ n}\end{aligned}$$

As we see, the last line is the transformation rule of a rank-2 mixed tensor from the unprimed system to the primed system. The tensorial nature of the covariant and contravariant types of the Kronecker delta can be established by the metric rules, i.e. all variance types of a tensor are tensors (see exercise 14 of § 2.2).
Note: the proof can be simply established by noting that the Kronecker delta is a metric tensor and hence by the result of the previous exercise it must be a tensor.

6. Why is the metric tensor symmetric?
Answer: From a formal (or superficial) perspective this can be easily justified by the fact that its elements are given by the inner product of two basis vectors (i.e. $g_{ij} = \mathbf{E}_i \cdot \mathbf{E}_j$ and $g^{ij} = \mathbf{E}^i \cdot \mathbf{E}^j$) and since inner product of vectors is commutative then the metric tensor should be symmetric. However, from a more substantial perspective let assume that the metric tensor is not symmetric (neither it is anti-symmetric) and hence it can be decomposed into a symmetric part $(g_{ij} + g_{ji})/2$ and an anti-symmetric part $(g_{ij} - g_{ji})/2$ (see the upcoming note). Now, if we note that the purpose of the metric tensor is to determine the geometry of the space through its involvement in the quadratic form [i.e. $(ds)^2 = g_{ij}dx^i dx^j$] then we can see that even if the metric tensor was (for the sake of argument) not symmetric it in effect is still like a symmetric tensor. The reason is that:

$$\begin{aligned}(ds)^2 &= g_{ij}dx^i dx^j \\ &= \left[\frac{(g_{ij}+g_{ji})}{2} + \frac{(g_{ij}-g_{ji})}{2}\right]dx^i dx^j \\ &= \frac{\left(g_{ij}dx^i dx^j + g_{ji}dx^i dx^j\right)}{2} + \frac{\left(g_{ij}dx^i dx^j - g_{ji}dx^i dx^j\right)}{2} \\ &= \frac{\left(g_{ij}dx^i dx^j + g_{ji}dx^i dx^j\right)}{2} + \frac{\left(g_{ij}dx^i dx^j - g_{ij}dx^j dx^i\right)}{2} \\ &= \frac{\left(g_{ij}dx^i dx^j + g_{ji}dx^i dx^j\right)}{2} + \frac{\left(g_{ij}dx^i dx^j - g_{ij}dx^i dx^j\right)}{2} \\ &= \frac{\left(g_{ij}dx^i dx^j + g_{ji}dx^i dx^j\right)}{2} + 0 \\ &= \frac{(g_{ij}+g_{ji})}{2}dx^i dx^j\end{aligned}$$

where in line 2 we decompose the metric tensor into its symmetric and anti-symmetric parts, and in line 4 we exchange the dummy indices in the fourth term. The last line means that even if the metric tensor is not symmetric, only its symmetric part is involved in determining the quadratic form (and hence the geometry) of the space. Therefore, we can always assume that the metric tensor behaves like a symmetric tensor. In brief, for all practical purposes the metric tensor can be treated as a symmetric tensor even if we cannot establish this fact theoretically and formally.[65]

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[65] Some of these generalizations may be questionable.

Note: any rank-2 tensor A_{ij} can be decomposed into a symmetric part $A_{(ij)}$ and an anti-symmetric part $A_{[ij]}$ where the following relations apply:

$$A_{ij} = A_{(ij)} + A_{[ij]} \qquad A_{(ij)} = \frac{1}{2}\left(A_{ij} + A_{ji}\right) \qquad A_{[ij]} = \frac{1}{2}\left(A_{ij} - A_{ji}\right)$$

If A_{ij} is symmetric then $A_{[ij]} = 0$ and hence $A_{ij} = A_{(ij)}$, and if A_{ij} is anti-symmetric then $A_{(ij)} = 0$ and hence $A_{ij} = A_{[ij]}$.

7. Summarize the main properties of the line element ds in Riemannian space highlighting its relation to the metric tensor.

Answer: We note the following:

• In Riemannian space, the line element ds is defined by $ds = \sqrt{g_{ij}dx^i dx^j}$ where g_{ij} is the metric tensor of the space.

• Line element represents the geometry of the space like the metric tensor itself (which is the only geometrically significant part in its definition).

• Line element is invariant under coordinate transformations (see next exercise). In fact, this represents the invariance of the geometric properties of the space (which are represented by the metric tensor) across different coordinate systems that coordinate the space since these properties are real and hence they should not depend on the employed coordinate systems (which have only conventional value).

8. Show the invariance of the quadratic form $(ds)^2$ across coordinate systems.

Answer: Let have an unprimed coordinate system with metric tensor g_{ij} and a primed coordinate system with metric tensor g'_{mn}. We have:

$$\begin{aligned}
(ds)^2 &= g_{ij}dx^i dx^j \\
&= g_{ij}\frac{\partial x^i}{\partial x'^m}\frac{\partial x^j}{\partial x'^n}dx'^m dx'^n \\
&= g'_{mn}dx'^m dx'^n \\
&= (ds')^2
\end{aligned}$$

where in line 2 we use the chain rule and in line 3 we use the transformation rule of rank-2 covariant tensor (since the metric g_{ij} is such a tensor; see exercise 4).

Note: we may argue more simply as follows:

$$\begin{aligned}
d\mathbf{r}\cdot d\mathbf{r} &= d\mathbf{r}'\cdot d\mathbf{r}' \\
(ds)^2 &= (ds')^2
\end{aligned}$$

where the first equality is justified by the fact that $d\mathbf{r} = d\mathbf{r}'$ since the infinitesimal position differential (or infinitesimal displacement) is an invariant geometric property of the space and hence it is independent of the coordinate system, while the second equality is based on the fundamental relation between ds and $d\mathbf{r}$.

9. Give the metric tensor of the 3D Euclidean space for the following coordinate systems: orthonormal Cartesian, cylindrical and spherical.

Answer: For orthonormal Cartesian coordinate system the metric tensor is given in its covariant and contravariant forms by the 3×3 unit matrix, that is:

$$[g_{ij}] = [\delta_{ij}] = \text{diag}\,[1,1,1] = \left[\delta^{ij}\right] = \left[g^{ij}\right]$$

For cylindrical coordinate system identified by the coordinates (ρ, ϕ, z), the metric tensor is given in its covariant and contravariant forms by:

$$[g_{ij}] = \text{diag}\left[1, \rho^2, 1\right] \qquad \left[g^{ij}\right] = \text{diag}\left[1, \rho^{-2}, 1\right]$$

For spherical coordinate system identified by the coordinates (r, θ, ϕ), the metric tensor is given in its covariant and contravariant forms by:

$$[g_{ij}] = \text{diag}\left[1, r^2, r^2\sin^2\theta\right] \qquad \left[g^{ij}\right] = \text{diag}\left[1, r^{-2}, r^{-2}\sin^{-2}\theta\right]$$

10. Give the metric tensor of the 4D flat spacetime of special relativity using the following 3D spatial coordinate systems: orthonormal Cartesian, cylindrical and spherical.
Answer: For orthonormal Cartesian coordinate system we have (see the upcoming note):
$$[g_{\mu\nu}] = \text{diag}\,[1, -1, -1, -1] = [g^{\mu\nu}]$$
For cylindrical coordinate system we have:
$$[g_{\mu\nu}] = \text{diag}\left[1, -1, -\rho^2, -1\right] \qquad\qquad [g^{\mu\nu}] = \text{diag}\left[1, -1, -\rho^{-2}, -1\right]$$
For spherical coordinate system we have:
$$[g_{\mu\nu}] = \text{diag}\left[1, -1, -r^2, -r^2 \sin^2\theta\right] \qquad\qquad [g^{\mu\nu}] = \text{diag}\left[1, -1, -r^{-2}, -r^{-2}\sin^{-2}\theta\right]$$
Note: the metric diag $[1, -1, -1, -1]$ is known as the Minkowski metric. The Minkowski metric may also be given by the following form: diag $[-1, 1, 1, 1]$. The Minkowski metric may also refer to the metric of the flat 4D spacetime in general regardless of using an underlying 3D spatial orthonormal Cartesian system or other 3D spatial systems like cylindrical and spherical (although in such cases we usually use the term "Lorentz metric" to distinguish it from the specific form diag $[1, -1, -1, -1]$). We should also note that the above Minkowski metric may also be symbolized with $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$ (as we usually do in other places in this book) to distinguish it from the metric of curved spacetime of general relativity which is commonly symbolized with $g_{\mu\nu}$ and $g^{\mu\nu}$.
11. Justify the use of the Kronecker delta as an index replacement operator.
Answer: Noting that the Kronecker delta is 1 when its indices are identical (i.e. $j = i$) and 0 when its indices are different (i.e. $j \neq i$), we have:
$$\begin{aligned} \delta^i_j A^j &= \delta^i_i A^i + 0 + \cdots + 0 = A^i \\ \delta^j_i A_j &= \delta^i_i A_i + 0 + \cdots + 0 = A_i \end{aligned}$$
where the summation convention (over j) is used. This similarly applies to any tensor of any rank and any variance type.
12. Show that the covariant and contravariant components of a vector $\mathbf{v}$ can be obtained from the relations: $\mathbf{v} \cdot \mathbf{E}_i = v_i$ and $\mathbf{v} \cdot \mathbf{E}^i = v^i$.
Answer:
$$\begin{aligned} \mathbf{v} \cdot \mathbf{E}_i &= \left(v_j \mathbf{E}^j\right) \cdot \mathbf{E}_i = v_j \left(\mathbf{E}^j \cdot \mathbf{E}_i\right) = v_j \delta^j_i = v_i \\ \mathbf{v} \cdot \mathbf{E}^i &= \left(v^j \mathbf{E}_j\right) \cdot \mathbf{E}^i = v^j \left(\mathbf{E}_j \cdot \mathbf{E}^i\right) = v^j \delta^i_j = v^i \end{aligned}$$
where in the last steps the Kronecker delta is used as an index replacement operator.
Note: this can be easily generalized to tensors of higher ranks. For example, if $\mathbf{V} = V_j{}^k \mathbf{E}^j \mathbf{E}_k$ then:
$$\begin{aligned} \mathbf{V} \cdot \mathbf{E}_i &= \left(V_j{}^k \mathbf{E}^j \mathbf{E}_k\right) \cdot \mathbf{E}_i = V_j{}^k \left(\mathbf{E}^j \cdot \mathbf{E}_i\right) \mathbf{E}_k = V_j{}^k \delta^j_i \mathbf{E}_k = V_i{}^k \mathbf{E}_k \\ \mathbf{V} \cdot \mathbf{E}^i &= \left(V_j{}^k \mathbf{E}^j \mathbf{E}_k\right) \cdot \mathbf{E}^i = V_j{}^k \mathbf{E}^j \left(\mathbf{E}_k \cdot \mathbf{E}^i\right) = V_j{}^k \mathbf{E}^j \delta^i_k = V_j{}^i \mathbf{E}^j \end{aligned}$$
13. Justify the use of the metric tensor as an index shifting operator.
Answer: Using the results of exercise 12, we have (with $\mathbf{A}$ being a vector):
$$\begin{aligned} A_i &= \mathbf{A} \cdot \mathbf{E}_i = A^j \mathbf{E}_j \cdot \mathbf{E}_i = A^j g_{ji} \\ A^i &= \mathbf{A} \cdot \mathbf{E}^i = A_j \mathbf{E}^j \cdot \mathbf{E}^i = A_j g^{ji} \end{aligned}$$
This similarly applies to any tensor of any rank and any variance type.
14. Show that in orthonormal Cartesian systems of Euclidean spaces there is no difference between the covariant and contravariant types of tensors.
Answer: We have:
$$A_i = g_{ij} A^j = \delta_{ij} A^j = \delta^i_j A^j = A^i$$

$$A^i \quad = \quad g^{ij}A_j = \delta^{ij}A_j = \delta_i^j A_j = A_i$$

where in the first steps we use the metric for index shifting, in the second steps we use the fact that the metric tensor in orthonormal Cartesian systems is the Kronecker delta, in the third steps we use $\delta_{ij} = \delta_j^i$ and $\delta^{ij} = \delta_i^j$ (see the upcoming note), and in the last steps we use the Kronecker delta for index replacement. This similarly applies to any tensor of any rank and any variance type.
Note: the equalities $\delta_{ij} = \delta_j^i$ and $\delta^{ij} = \delta_i^j$ (which are used in the third steps) are justified by the fact that the corresponding components of all the variance types of the Kronecker delta are the same (i.e. $\delta_{ij} = \delta^{ij} = \delta_j^i$) where the "corresponding components" mean components that have the same indices (e.g. $\delta_{11} = \delta^{11} = \delta_1^1 = 1$ and $\delta_{12} = \delta^{12} = \delta_2^1 = 0$).

15. Show that the metric tensor is diagonal *iff* the coordinate system is orthogonal.
Answer: We have two parts:
• If the metric tensor is diagonal then the coordinate system is orthogonal: this can be easily inferred from the relation $g_{ij} = \mathbf{E}_i \cdot \mathbf{E}_j$ because when the metric tensor is diagonal then we have $g_{ij} = 0$ $(i \neq j)$ and hence $\mathbf{E}_i$ and $\mathbf{E}_j$ must be orthogonal since their inner product is zero (noting that $\mathbf{E}_i$ and $\mathbf{E}_j$ cannot be zero although even in this case they can be considered orthogonal).
• If the coordinate system is orthogonal then the metric tensor is diagonal: this can also be inferred from the relation $g_{ij} = \mathbf{E}_i \cdot \mathbf{E}_j$ because when the coordinate system is orthogonal then we have $\mathbf{E}_i \cdot \mathbf{E}_j = 0$ $(i \neq j)$ and hence $g_{ij} = 0$ $(i \neq j)$ which means that the metric tensor is diagonal.
Note: when g_{ij} is diagonal then g^{ij} (which is its inverse) should also be diagonal (and vice versa) and hence the above statement should similarly apply (in fact the covariant basis vectors are orthogonal *iff* the contravariant basis vectors are orthogonal). The proof can also be repeated on g^{ij} following similar arguments as above and using $g^{ij} = \mathbf{E}^i \cdot \mathbf{E}^j$. Regarding the mixed form g_j^i we have $g_j^i = \mathbf{E}^i \cdot \mathbf{E}_j = \delta_j^i$ and hence the above statement is always valid although locally (i.e. with respect to the local Cartesian system regardless of the orthogonality of the employed system globally) since δ_j^i is diagonal. Yes, if the metric tensor is represented globally by the Kronecker delta then the coordinate system should be orthonormal Cartesian and hence the above statement is also valid. We should finally note that "diagonal" and "orthogonal" in the above statement are commonly meant to be global although the statement should also be valid locally.
16. Show that the constancy of the metric tensor is a sufficient but not necessary condition for the space to be flat.[66]
Answer: We have two parts to prove:
• If the metric tensor is constant then the space is flat (i.e. being sufficient condition): the metric tensor is invertible (since we always have $g^{ik}g_{kj} = g_{jk}g^{ki} = \delta_j^i$), and therefore it is always possible to transform the metric tensor to the identity tensor (or the Kronecker delta tensor). Now, in general Riemannian space this transformation applies only locally (inline with the fact that Riemannian space is locally flat). However, the constancy of the metric tensor over the entire space means that such a transformation is valid globally which means that we can transform the metric tensor over the entire space to the identity tensor. Now, the identity tensor (as a global metric tensor) is equivalent to having a global orthonormal Cartesian coordinate system, and this means that the space is flat (according to our main criterion for space flatness which is the possibility of coordination by a Cartesian system globally; see § 2.3).
• When the space is flat the metric tensor may not be constant (i.e. not being necessary condition): this should be obvious because we have examples of non-constant metric tensors of flat spaces, e.g. the metric tensor of 3D Euclidean space coordinated by a spherical coordinate system since this metric is obviously coordinate-dependent and hence it is not constant although the space is flat (see exercises 9 and 10).
Note: the first part (i.e. the "If" part) may also be established by the fact that the constancy of the metric tensor is equivalent to the vanishing of the Christoffel symbols (see exercise 7 of § 2.6) and this vanishing is equivalent to having Cartesian (or at least rectilinear) coordinate system (see exercise 6 of § 2.6) which is sufficient for establishing space flatness according to our main criterion.

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[66] As indicated earlier, the constancy of the metric tensor means the constancy of its individual elements.

2.5.1 Metric Tensor of 4D Spacetime of General Relativity

1. What we mean by "general form" when we say "the metric tensor of the flat spacetime of special relativity has a general form"?
   Answer: We mean by "general form" a specific and unique form that applies in all cases such as diag $[1, -1, -1, -1]$ and diag $\left[1, -1, -r^2, -r^2 \sin^2 \theta\right]$ which represent the metric of any spacetime of special relativity unlike the metric tensor of general relativity which cannot be given in such a specific and unique form that applies in all cases since it depends on the physical setting (i.e. the distribution and flow of matter and energy).
2. Compare the metrics of special relativity and general relativity.
   Answer: We note the following:
   • The spacetime of special relativity is flat while the spacetime of general relativity is curved. Accordingly, the metric of special relativity can be constant (i.e. independent of coordinates as it is the case with the Minkowski metric diag $[1, -1, -1, -1]$) while the metric of general relativity is generally variable (except in some simple or degenerate cases).
   • The metric of special relativity has general form while the metric of general relativity depends on the case according to the particular physical setting and presumptions. In other words, the metric of special relativity represents a single geometry while the metric of general relativity represents a geometry that depends on the particular physical setting. So, all the different forms of the metric tensor of special relativity represent the same geometry of the spacetime. This arises from the fact that flat geometry is unique (i.e. we have only one flat geometry) while curved geometry is not unique (i.e. we have infinitely many curved geometries although all these geometries are generically labeled as curved in contrast to flat).
   Note: the uniqueness of flat geometry should be subject to certain (but rather minor) restrictions such as the signature of the space. However, as we are concerned only with the spacetimes of the relativity theories, this should not impose any real restriction (and hence our assertion of uniqueness is justified).
3. What is the implication of not having a unique and general form of the metric in general relativity?
   Answer: The main implication is that the results obtained from solving general relativistic problems are metric-specific and hence they are of limited validity. In fact, some of the results may even be coordinate-specific and hence their validity is more limited (see § 10.1.7).

2.6 Christoffel Symbols

1. The Christoffel symbols are not tensors although they have indices that refer to coordinates and basis vectors like tensors. Justify.
   Answer: Tensors are characterized by their transformation rules and not by their indices. In other words, non-scalar tensors must have indices but not all mathematical objects that have indices are tensors (also see the next exercise).
2. Explain "affine tensors" within the context of Christoffel symbols.
   Answer: Affine tensors are mathematical entities that are similar in structure to tensors by having indices that refer to coordinates and basis vectors. Moreover, they transform invariantly across all affine coordinate systems and hence they behave as tensors in these systems. However, they are not tensors unconditionally because they do not transform invariantly across all types of coordinate systems that include non-affine systems. For example, Γ^k_{ij} has 3 indices that refer to space coordinates and basis vectors and hence it looks like a mixed tensor of type $(1, 2)$. Moreover, Γ^k_{ij} transforms invariantly across all affine coordinate systems since it is identically zero in all these systems.[67] However, it does not transform invariantly across coordinate systems that include non-affine systems. For example, Γ^k_{ij} is identically zero in rectilinear Cartesian systems but it is not identically zero in curvilinear spherical systems and hence when it transforms between these two types of system it does not satisfy

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[67] In fact, being identically zero in all these systems is because it transforms formally like a tensor in these systems (see exercise 12; also refer to question 5.3 of B3X).

the invariance property of the zero tensor, i.e. the zero tensor must remain zero under permissible coordinate transformations (see § 2.2).

3. What are the number of Christoffel symbols and the number of independent Christoffel symbols of each kind in the spacetime of general relativity?
Answer: The number of Christoffel symbols and the number of independent Christoffel symbols in an nD space are given respectively by:

$$n^3 \qquad \text{and} \qquad \frac{n^2(n+1)}{2}$$

Now, the spacetime of general relativity is 4D and hence we have:

$$n^3 = 64 \qquad \text{and} \qquad \frac{n^2(n+1)}{2} = 40$$

4. What are the physical dimensions of the Christoffel symbols of the first and second kind?
Answer: The Christoffel symbols of the first and second kind are given by:

$$\begin{aligned}[ij,l] &= \frac{1}{2}(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij}) \\ \Gamma^k_{ij} &= \frac{g^{kl}}{2}(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij})\end{aligned}$$

Now, if we note that the metric tensor is dimensionless (see § 2.5) and the dimensions of the differential operator (i.e. $\partial_i \equiv \frac{\partial}{\partial x^i}$) are reciprocal length then we can conclude that the physical dimensions of the Christoffel symbols are reciprocal length (e.g. m^{-1}).
Note: as indicated earlier (see § 2.5), the metric tensor is not necessarily dimensionless and the dimensions of the differential operator are not necessarily reciprocal length and hence the physical dimensions of the Christoffel symbols are not necessarily reciprocal length. However, when obtaining other dimensional relations that have definite physical dimensionality and are based on the dimensionality of the Christoffel symbols we can assume the validity of the obtained result (i.e. the physical dimensions of the Christoffel symbols are reciprocal length) by assuming a certain type of coordinate system whose metric tensor is dimensionless and a certain type of coordinates whose physical dimensionality is length. The generality of the obtained dimensional results will then be guaranteed by the definite nature of the physical dimensionality of the given quantity and the required consistency in the physical dimensions of any particular physical quantity. In other words, the combination of dimensional and non-dimensional quantities involved in the definition of the particular physical quantity will ensure the correct physical dimensionality of this quantity.

5. Prove the following relation: $\partial_j g_{il} = [ij,l] + [lj,i]$.
Answer: This was proved in problem 4.

6. Show that the Christoffel symbols vanish identically *iff* the coordinate system is rectilinear (and Cartesian in particular).
Answer: This can be easily inferred from the fundamental definition of the Christoffel symbols (i.e. $\partial_j \mathbf{E}_i = \Gamma^k_{ij}\mathbf{E}_k$ and its alike) because:
• If $\Gamma^k_{ij} = 0$ then $\partial_j \mathbf{E}_i = \mathbf{0}$ (which means the basis vectors are constants) and hence the system is rectilinear.
• If the system is rectilinear then the basis vectors are constants (which means $\partial_j \mathbf{E}_i = \mathbf{0}$) and hence $\Gamma^k_{ij} = 0$ (since $\mathbf{E}_k \neq \mathbf{0}$).

7. Prove that in any coordinate system, all the Christoffel symbols of either kind vanish identically *iff* all the components of the metric tensor in the given coordinate system are constants.[68]
Answer: The proof is made of two parts (i.e. the "if" part and the "only if" part):
• The "if" part: the Christoffel symbols are given by.

$$[ij,k] = \frac{1}{2}(\partial_j g_{ik} + \partial_i g_{jk} - \partial_k g_{ij})$$

---

[68] "Identically" should indicate the validity over the entire space. However, the statement may also apply in a given region.

$$\Gamma_{ij}^{k} = \frac{g^{kl}}{2}\left(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij}\right)$$

So, both kinds of the Christoffel symbols are sum of terms containing partial derivatives of components of the metric tensor. Therefore, if all the components of the metric tensor are constants then all these partial derivatives will vanish identically and hence the Christoffel symbols will also vanish identically. This similarly applies to the contravariant metric tensor because its constancy implies the constancy of the covariant metric tensor (and vice versa) because they are inverses of each other.

• The "only if" part: according to the relation $\partial_j g_{il} = [ij, l] + [lj, i]$ which we proved earlier, if the Christoffel symbols vanish identically then $\partial_j g_{il} = 0$ and hence the metric coefficients g_{il} are constants. This similarly applies to the Christoffel symbols of the second kind because if they vanish then the Christoffel symbols of the first kind (which are used in the aforementioned relation) must also vanish because the two kinds are linked by the index shifting operation which does not change the zero.

Note 1: we may also prove the "only if" part by arguing that when the Christoffel symbols vanish identically the covariant derivative (see § 2.7) becomes partial derivative. Now, by the Ricci theorem[69] the covariant derivative of the metric tensor is zero and hence the partial derivative is zero in this case. Accordingly, the metric tensor must be constant. The "only if" part (and even the "if" part) may also be proved by using the result of exercise 6 because if the Christoffel symbols vanish identically then the system is rectilinear and hence the metric tensor must be constant (as can be concluded from the relations $g_{ij} = \mathbf{E}_i \cdot \mathbf{E}_j$ and $g^{ij} = \mathbf{E}^i \cdot \mathbf{E}^j$).[70]

Note 2: for more clarity, in the generalization of the above proof we used two simple facts: (a) the covariant metric tensor is constant *iff* the contravariant metric tensor is constant (due to their inverse relation), and (b) the Christoffel symbols of the first kind vanish *iff* the Christoffel symbols of the second kind vanish (due to their metric relation).

8. Show that the vanishing of the Christoffel symbols identically is a sufficient but not necessary condition for space flatness.

   Answer: We have two parts to prove:

   • Being sufficient condition: if the Christoffel symbols vanish identically then according to the result of exercise 6 the system is Cartesian (or rectilinear) and hence the space is flat according to the main criterion for flatness (see § 2.3). Alternatively, if the Christoffel symbols vanish identically then the metric is constant according to exercise 7, and hence the space is flat according to exercise 16 of § 2.5.

   • Being not necessary condition: if the space is flat and it is coordinated by a curvilinear coordinate system (like cylindrical or spherical in 3D Euclidean space) the Christoffel symbols do not vanish (although some might vanish), i.e. vanishing of the Christoffel symbols is not a necessary condition for space flatness.

9. Summarize the main properties of the Christoffel symbols.

   Answer: We note the following:

   • They are rank-3 affine tensors but they are not general tensors.

   • They are intrinsic to the space since they solely depend on the metric.

   • They vanish identically in rectilinear systems.

   • They vanish identically *iff* the metric coefficients are constants.

   • They have two types (first and second) linked through the index shifting operators.

   • We defined them as being representatives of the components of the partial derivative of the basis vectors with respect to the coordinates in the direction of the basis vectors themselves, i.e. $\partial_j \mathbf{E}_i = \Gamma_{ij}^{k} \mathbf{E}_k$ and $\partial_j \mathbf{E}^i = -\Gamma_{kj}^{i} \mathbf{E}^k$ (although they can be defined in other ways).

   • They are essential elements in the tensorial description of the geometry of multi-dimensional curved spaces and hence they enter in almost all the definitions, operations and procedures in these spaces (e.g. definition of curvature tensors like Riemann-Christoffel and Ricci curvature tensors, parallel transport, and tensor differentiation). They are also used in the formulations of flat spaces when these spaces are coordinated by curvilinear systems.

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[69] The Ricci theorem states that the covariant derivative of the metric tensor is identically zero.

[70] In fact, there are many different ways and routes for proving the above statements and their alike (although vigilance is required to avoid circularity).

- In nD space, there are n^3 Christoffel symbols of either kind but only $\left[n^2(n+1)\right]/2$ of these are independent.

10. Show that $\mathbf{E}^k \cdot \partial_j \mathbf{E}_i = \Gamma^k_{ij}$ and $\mathbf{E}_k \cdot \partial_j \mathbf{E}^i = -\Gamma^i_{kj}$.
Answer: We have:

$$\begin{aligned}
\partial_j \mathbf{E}_i &= \Gamma^a_{ij} \mathbf{E}_a \\
\mathbf{E}^k \cdot \partial_j \mathbf{E}_i &= \Gamma^a_{ij} \mathbf{E}^k \cdot \mathbf{E}_a \\
\mathbf{E}^k \cdot \partial_j \mathbf{E}_i &= \Gamma^a_{ij} \delta^k_a \\
\mathbf{E}^k \cdot \partial_j \mathbf{E}_i &= \Gamma^k_{ij}
\end{aligned}$$

where line 1 is from the fundamental definition of the Christoffel symbols, in line 2 we inner-multiply both sides with $\mathbf{E}^k$, in line 3 we use $\mathbf{E}^k \cdot \mathbf{E}_a = \delta^k_a$ and in line 4 we use δ^k_a for index replacement.
Similarly:

$$\begin{aligned}
\partial_j \mathbf{E}^i &= -\Gamma^i_{aj} \mathbf{E}^a \\
\mathbf{E}_k \cdot \partial_j \mathbf{E}^i &= -\Gamma^i_{aj} \mathbf{E}_k \cdot \mathbf{E}^a \\
\mathbf{E}_k \cdot \partial_j \mathbf{E}^i &= -\Gamma^i_{aj} \delta^a_k \\
\mathbf{E}_k \cdot \partial_j \mathbf{E}^i &= -\Gamma^i_{kj}
\end{aligned}$$

11. Show that:

$$\Gamma^k_{ij} = \frac{\partial x^k}{\partial x_a} \frac{\partial^2 x_a}{\partial x^i \partial x^j}$$

where the lower-indexed x are Cartesian coordinates and the upper-indexed x are general coordinates.
Answer: From the previous question we have:

$$\begin{aligned}
\Gamma^k_{ij} &= \mathbf{E}^k \cdot \partial_j \mathbf{E}_i \\
&= \left(\nabla x^k\right) \cdot \frac{\partial}{\partial x^j}\left(\frac{\partial \mathbf{r}}{\partial x^i}\right) \\
&= \left(\frac{\partial x^k}{\partial x_a}\right) \frac{\partial}{\partial x^j}\left(\frac{\partial x_a}{\partial x^i}\right) \\
&= \frac{\partial x^k}{\partial x_a} \frac{\partial^2 x_a}{\partial x^i \partial x^j}
\end{aligned}$$

where in line 2 we use the definitions of $\mathbf{E}^k$, ∂_j and $\mathbf{E}_i$, in line 3 we replace the vector (or symbolic) notation with the indicial notation, and in line 4 we simplify and use the commutativity of the partial differential operators.
Note: to be more clear we have:

$$\left(\nabla x^k\right) \cdot \frac{\partial}{\partial x^j}\left(\frac{\partial \mathbf{r}}{\partial x^i}\right) = \delta_{ab}\left(\frac{\partial x^k}{\partial x_a}\right) \frac{\partial}{\partial x^j}\left(\frac{\partial x_b}{\partial x^i}\right) = \left(\frac{\partial x^k}{\partial x_a}\right) \frac{\partial}{\partial x^j}\left(\frac{\partial x_a}{\partial x^i}\right) = \frac{\partial x^k}{\partial x_a} \frac{\partial^2 x_a}{\partial x^i \partial x^j}$$

where in the first equality we use the definition of inner product in Cartesian coordinates (noting that δ_{ab} is the metric of the Cartesian system and ∇x^k may be written loosely as $\frac{\partial x^k}{\partial \mathbf{r}}$).

12. Show that the Christoffel symbols transform as:

$$\Gamma'^k_{ij} = \frac{\partial x'^k}{\partial x^b} \frac{\partial x^d}{\partial x'^i} \frac{\partial x^c}{\partial x'^j} \Gamma^b_{dc} + \frac{\partial x'^k}{\partial x^c} \frac{\partial^2 x^c}{\partial x'^i \partial x'^j}$$

Hence, conclude that the Christoffel symbols are affine tensors but not general tensors.
Answer: If we transform from an unprimed system O to a primed system O' (using unprimed and primed symbols in these systems respectively) then we have:

$$\Gamma'^k_{ij} = \frac{\partial x'^k}{\partial x_a} \frac{\partial^2 x_a}{\partial x'^i \partial x'^j}$$

$$
\begin{aligned}
&= \frac{\partial x'^k}{\partial x_a}\frac{\partial}{\partial x'^i}\left(\frac{\partial x_a}{\partial x'^j}\right)\\
&= \frac{\partial x'^k}{\partial x^b}\frac{\partial x^b}{\partial x_a}\frac{\partial}{\partial x'^i}\left(\frac{\partial x_a}{\partial x^c}\frac{\partial x^c}{\partial x'^j}\right)\\
&= \frac{\partial x'^k}{\partial x^b}\frac{\partial x^b}{\partial x_a}\left[\frac{\partial x^c}{\partial x'^j}\frac{\partial}{\partial x'^i}\left(\frac{\partial x_a}{\partial x^c}\right)+\frac{\partial x_a}{\partial x^c}\frac{\partial}{\partial x'^i}\left(\frac{\partial x^c}{\partial x'^j}\right)\right]\\
&= \frac{\partial x'^k}{\partial x^b}\frac{\partial x^b}{\partial x_a}\left[\frac{\partial x^c}{\partial x'^j}\frac{\partial}{\partial x^d}\left(\frac{\partial x_a}{\partial x^c}\right)\frac{\partial x^d}{\partial x'^i}+\frac{\partial x_a}{\partial x^c}\frac{\partial}{\partial x'^i}\left(\frac{\partial x^c}{\partial x'^j}\right)\right]\\
&= \frac{\partial x'^k}{\partial x^b}\frac{\partial x^b}{\partial x_a}\left[\frac{\partial x^c}{\partial x'^j}\frac{\partial x^d}{\partial x'^i}\frac{\partial^2 x_a}{\partial x^d\partial x^c}+\frac{\partial x_a}{\partial x^c}\frac{\partial^2 x^c}{\partial x'^i\partial x'^j}\right]\\
&= \frac{\partial x'^k}{\partial x^b}\frac{\partial x^d}{\partial x'^i}\frac{\partial x^c}{\partial x'^j}\left(\frac{\partial x^b}{\partial x_a}\frac{\partial^2 x_a}{\partial x^d\partial x^c}\right)+\frac{\partial x'^k}{\partial x^b}\frac{\partial x^b}{\partial x_a}\frac{\partial x_a}{\partial x^c}\frac{\partial^2 x^c}{\partial x'^i\partial x'^j}\\
&= \frac{\partial x'^k}{\partial x^b}\frac{\partial x^d}{\partial x'^i}\frac{\partial x^c}{\partial x'^j}\Gamma^b_{dc}+\frac{\partial x'^k}{\partial x^b}\frac{\partial x^b}{\partial x_a}\frac{\partial x_a}{\partial x^c}\frac{\partial^2 x^c}{\partial x'^i\partial x'^j}\\
&= \frac{\partial x'^k}{\partial x^b}\frac{\partial x^d}{\partial x'^i}\frac{\partial x^c}{\partial x'^j}\Gamma^b_{dc}+\frac{\partial x'^k}{\partial x^c}\frac{\partial^2 x^c}{\partial x'^i\partial x'^j} \qquad (8)
\end{aligned}
$$

where line 1 is from the previous question, line 2 is a matter of notation, line 3 is the chain rule (twice), line 4 is the product rule, line 5 is the chain rule, lines 6 and 7 are a matter of notation and ordering, line 8 is from the previous question, and line 9 is the chain rule. As we see, the first term of the last line is the expected transformation rule if Γ^b_{dc} is a tensor but the second term (which does not vanish in general coordinate systems) spoils this and hence in general coordinate systems Γ^b_{dc} does not transform as a tensor and hence it is not a general tensor. However, the second term vanishes in affine systems (because $\frac{\partial x^c}{\partial x'^j}$ is constant and $\frac{\partial^2 x^c}{\partial x'^i\partial x'^j}=0$) and hence in affine systems Γ^b_{dc} does transform as a tensor and hence it is an affine tensor.

2.7 Tensor Differentiation

1. What is the essence of tensor differentiation?
   Answer: Tensor differentiation (i.e. covariant and intrinsic differentiation) is no more than an ordinary differentiation (i.e. partial and total differentiation) in which the variation of the basis vectors, as well as the variation of the components, is taken into account. In other words, tensor differentiation is a generalization of the ordinary differentiation in rectilinear systems (where the basis vectors are constants and hence their derivative vanishes) to the differentiation in curvilinear systems (where the basis vectors are variables and hence their derivative does not vanish).
2. Show that the covariant and intrinsic derivatives of scalars and affine tensors of higher ranks are the same as the ordinary derivatives (i.e. partial and total derivatives).
   Answer: The covariant derivative is given by:

$$
\begin{aligned}
A^{i_1\ldots i_m}_{j_1\ldots j_n;k} &= \partial_k A^{i_1\ldots i_m}_{j_1\ldots j_n}+\Gamma^{i_1}_{lk}A^{l\ldots i_m}_{j_1\ldots j_n}+\cdots+\Gamma^{i_m}_{lk}A^{i_1\ldots l}_{j_1\ldots j_n}\\
&\quad -\Gamma^{l}_{j_1k}A^{i_1\ldots i_m}_{l\ldots j_n}-\cdots-\Gamma^{l}_{j_nk}A^{i_1\ldots i_m}_{j_1\ldots l}
\end{aligned}
$$

   Now, scalars have no indices and hence there are no Christoffel symbol terms, while in affine systems the Christoffel symbols are zero (see exercise 6 of § 2.6) and hence the Christoffel symbol terms vanish. So, in both cases only the partial derivative term remains and hence the covariant derivative becomes ordinary partial derivative.
   Regarding the intrinsic derivative, it is an inner product of the covariant derivative (which is partial derivative in this case) and the tangent vector and hence by the chain rule it becomes ordinary total derivative, i.e. $\frac{\partial A^{i_1\cdots i_m}_{j_1\cdots j_n}}{\partial x^k}\frac{dx^k}{dt}=\frac{dA^{i_1\cdots i_m}_{j_1\cdots j_n}}{dt}$.
   In fact, all these can be concluded more directly from the generic definitions of the covariant and

intrinsic derivatives [i.e. $\partial_k \left(A^{i\cdots m}_{j\cdots n}\mathbf{E}_i\cdots\mathbf{E}_m\mathbf{E}^j\cdots\mathbf{E}^n\right)$ and $\frac{d}{dt}\left(A^{i\cdots m}_{j\cdots n}\mathbf{E}_i\cdots\mathbf{E}_m\mathbf{E}^j\cdots\mathbf{E}^n\right)$] by noting that scalars are not associated with basis vectors while affine tensors are associated with constant basis vectors.

3. Show that the partial derivative of a scalar is a tensor (i.e. rank-1 tensor which is vector).
Answer: Scalar is a tensor (i.e. of rank-0) since it is invariant across all coordinate systems. Hence, if ϕ is a scalar and we have two coordinate systems (say primed O' and unprimed O) then we should have $\phi = \phi'$ (where the prime is a reference to O'). Now, if the partial derivative of the scalar is $\partial\phi/\partial x^i$ in O and is $\partial\phi'/\partial x'^j$ in O' then we have:

$$\frac{\partial\phi'}{\partial x'^j} = \frac{\partial\phi}{\partial x'^j} = \frac{\partial\phi}{\partial x^i}\frac{\partial x^i}{\partial x'^j} = \frac{\partial x^i}{\partial x'^j}\left(\frac{\partial\phi}{\partial x^i}\right)$$

where we used $\phi = \phi'$ in the first equality and the chain rule in the second equality. As we see, the last equality shows that the partial derivative transforms like a tensor and hence it is a tensor.
Note: it may be claimed that the proof can also be established by the result of exercise 2 because since the partial derivative of a scalar is the same as its covariant derivative then the partial derivative of a scalar is a tensor (since the covariant derivative is a tensor). However, this should lead to circularity because the tensorial nature of the covariant derivative must be established first by showing its tensorial nature (as we did in exercise 7 using the result of this exercise).

4. Show that the total derivative of a scalar along a t-parameterized curve is a tensor (i.e. scalar or rank-0 tensor).
Answer: If $\phi(t)$ is a scalar function along a t-parameterized curve $x^i(t)$ then by the chain rule we have:

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial x^i}\frac{dx^i}{dt}$$

Now, both $\frac{\partial\phi}{\partial x^i}$ and $\frac{dx^i}{dt}$ are vectors (i.e. rank-1 tensors of covariant and contravariant type respectively) and hence their contraction is also a tensor, i.e. rank-0 tensor (see exercise 9 of § 2.2). The tensorial nature of $\frac{\partial\phi}{\partial x^i}$ was shown in the previous exercise while the tensorial nature of $\frac{dx^i}{dt}$ is because it is the tangent vector to the curve (and hence it is a tensor by a geometric argument).
Note: it may be claimed that the tensorial nature of the total derivative of a scalar may also be inferred from the fact that the absolute derivative of a scalar is the same as its ordinary total derivative (i.e. $\frac{\delta\phi}{\delta t} \equiv \frac{d\phi}{dt}$; see exercise 2) plus the fact that $\frac{\delta\phi}{\delta t}$ is a tensor. However, this should lead to circularity as in the case of covariant derivative (see the note of exercise 3). In fact, the above demonstration should be used to demonstrate that $\frac{\delta\phi}{\delta t}$ is a tensor, i.e. $\frac{\delta\phi}{\delta t} \equiv \frac{d\phi}{dt} = \frac{\partial\phi}{\partial x^i}\frac{dx^i}{dt}$.

5. Demonstrate that the partial derivative of a non-scalar tensor is not a tensor (in general).
Answer: Let use a vector A_i (which is a rank-1 tensor) for this demonstration. This should be enough because if the partial derivative of A_i is not a tensor then the partial derivative of a tensor is not a tensor since the violation in this case (at least) means it is not valid in general. The strategy of this demonstration is to show that the partial derivative $A_{i,j}$ of A_i does not transform as a tensor. Now, let assume that $A_{i,j}$ in system O is transformed to $A'_{k,l}$ in system O'. Hence, we have:

$$\begin{aligned}
A'_{k,l} &\equiv \frac{\partial A'_k}{\partial x'^l} \\
&= \frac{\partial}{\partial x'^l}\left(\frac{\partial x^i}{\partial x'^k}A_i\right) \\
&= \frac{\partial A_i}{\partial x'^l}\frac{\partial x^i}{\partial x'^k} + \frac{\partial^2 x^i}{\partial x'^l\partial x'^k}A_i \\
&= \frac{\partial A_i}{\partial x^j}\frac{\partial x^j}{\partial x'^l}\frac{\partial x^i}{\partial x'^k} + \frac{\partial^2 x^i}{\partial x'^l\partial x'^k}A_i \\
&= \frac{\partial x^i}{\partial x'^k}\frac{\partial x^j}{\partial x'^l}A_{i,j} + \frac{\partial^2 x^i}{\partial x'^l\partial x'^k}A_i \qquad (9)
\end{aligned}$$

where in line 2 we use the transformation of A_i to A'_k (since A_i is a tensor), line 3 is the product rule, and line 4 is the chain rule. As we see, the first term in the last line is the same as the transformation rule of a tensor and therefore if we have only this term then $A_{i,j}$ transforms as a tensor and hence it is a tensor. However, the presence of the second term spoils this tensor-like transformation and therefore $A_{i,j}$ does not transform as a tensor and hence it is not a tensor.

Note: in affine systems the second term vanishes (because $\frac{\partial x^i}{\partial x'^k}$ is constant and $\frac{\partial^2 x^i}{\partial x'^l \partial x'^k} = 0$) and hence the partial derivative transforms as a tensor. This is inline with the fact that in affine systems the covariant derivative is the same as the partial derivative (i.e. partial derivative is affine tensor) as seen in exercise 2.

6. Demonstrate that the total derivative of a non-scalar tensor is not a tensor.

Answer: Let use A_i again for this demonstration (employing the same justification as in the previous exercise). So, according to the chain rule we have:

$$\frac{dA_i}{dt} = \frac{\partial A_i}{\partial x^j}\frac{dx^j}{dt} = A_{i,j}\frac{dx^j}{dt}$$

Now, since $A_{i,j}$ is not a tensor (as shown in the previous exercise) then $\frac{dA_i}{dt}$ is not a tensor. This is justified by the quotient rule of tensors because $\frac{dx^j}{dt}$ is a tensor so if $\frac{dA_i}{dt}$ is a tensor then by the quotient rule $A_{i,j}$ should be a tensor (which is not according to the previous exercise).

Note: we can present the quotient rule argument in a more formal way as follows:
According to the quotient rule, if ($\frac{dx^j}{dt}$ is a tensor and $\frac{dA_i}{dt}$ is a tensor) then ($A_{i,j}$ is a tensor).
But ($A_{i,j}$ is a tensor) is untrue, therefore ($\frac{dx^j}{dt}$ is a tensor and $\frac{dA_i}{dt}$ is a tensor) is untrue.
Now, since ($\frac{dx^j}{dt}$ is a tensor) is true then ($\frac{dA_i}{dt}$ is a tensor) is untrue, i.e. $\frac{dA_i}{dt}$ is not a tensor.

7. Show that the covariant derivative of a tensor is a tensor.

Answer: A full proof is lengthy, so what we will do is to demonstrate (using a covariant vector as an example) that the covariant derivative of vectors transforms as a tensor and hence it is a tensor. We then generalize by appealing to the pattern (as well as using the results of the previous exercises).
Let $A'_{a;l}$ be the covariant derivative of a covariant vector $\mathbf{A}$ in system O' and $A_{i;j}$ be the covariant derivative of $\mathbf{A}$ in system O, and hence we have:

$$\begin{aligned}
A'_{a;l} &= A'_{a,l} - \Gamma'^{k}_{al}A'_k \\
&= \left(\frac{\partial x^i}{\partial x'^a}\frac{\partial x^j}{\partial x'^l}A_{i,j} + \frac{\partial^2 x^i}{\partial x'^l \partial x'^a}A_i\right) - \left(\frac{\partial x'^k}{\partial x^b}\frac{\partial x^d}{\partial x'^a}\frac{\partial x^c}{\partial x'^l}\Gamma^b_{dc} + \frac{\partial x'^k}{\partial x^c}\frac{\partial^2 x^c}{\partial x'^a \partial x'^l}\right)\left(\frac{\partial x^i}{\partial x'^k}A_i\right) \\
&= \left(\frac{\partial x^i}{\partial x'^a}\frac{\partial x^j}{\partial x'^l}A_{i,j} + \frac{\partial^2 x^i}{\partial x'^l \partial x'^a}A_i\right) - \left(\frac{\partial x'^k}{\partial x^b}\frac{\partial x^d}{\partial x'^a}\frac{\partial x^c}{\partial x'^l}\Gamma^b_{dc}\frac{\partial x^i}{\partial x'^k}A_i + \frac{\partial x'^k}{\partial x^c}\frac{\partial^2 x^c}{\partial x'^a \partial x'^l}\frac{\partial x^i}{\partial x'^k}A_i\right) \\
&= \left(\frac{\partial x^i}{\partial x'^a}\frac{\partial x^j}{\partial x'^l}A_{i,j} + \frac{\partial^2 x^i}{\partial x'^l \partial x'^a}A_i\right) - \left(\frac{\partial x^i}{\partial x^b}\frac{\partial x^d}{\partial x'^a}\frac{\partial x^c}{\partial x'^l}\Gamma^b_{dc}A_i + \frac{\partial x^i}{\partial x^c}\frac{\partial^2 x^c}{\partial x'^a \partial x'^l}A_i\right) \\
&= \left(\frac{\partial x^i}{\partial x'^a}\frac{\partial x^j}{\partial x'^l}A_{i,j} + \frac{\partial^2 x^i}{\partial x'^l \partial x'^a}A_i\right) - \left(\delta^i_b\frac{\partial x^d}{\partial x'^a}\frac{\partial x^c}{\partial x'^l}\Gamma^b_{dc}A_i + \delta^i_c\frac{\partial^2 x^c}{\partial x'^a \partial x'^l}A_i\right) \\
&= \left(\frac{\partial x^i}{\partial x'^a}\frac{\partial x^j}{\partial x'^l}A_{i,j} + \frac{\partial^2 x^i}{\partial x'^l \partial x'^a}A_i\right) - \left(\frac{\partial x^d}{\partial x'^a}\frac{\partial x^c}{\partial x'^l}\Gamma^i_{dc}A_i + \frac{\partial^2 x^i}{\partial x'^a \partial x'^l}A_i\right) \\
&= \frac{\partial x^i}{\partial x'^a}\frac{\partial x^j}{\partial x'^l}A_{i,j} - \frac{\partial x^d}{\partial x'^a}\frac{\partial x^c}{\partial x'^l}\Gamma^i_{dc}A_i \\
&= \frac{\partial x^i}{\partial x'^a}\frac{\partial x^j}{\partial x'^l}A_{i,j} - \frac{\partial x^i}{\partial x'^a}\frac{\partial x^j}{\partial x'^l}\Gamma^d_{ij}A_d \\
&= \frac{\partial x^i}{\partial x'^a}\frac{\partial x^j}{\partial x'^l}\left(A_{i,j} - \Gamma^d_{ij}A_d\right) \\
&= \frac{\partial x^i}{\partial x'^a}\frac{\partial x^j}{\partial x'^l}A_{i;j}
\end{aligned}$$

where in line 1 we use the covariant derivative rule, in line 2 we substitute for $A'_{a,l}$, Γ'^{k}_{al} and A'_k using their transformation rules (see Eqs. 9, 8 and 5), in line 4 we use the chain rule, in line 6 we use Kronecker delta for index replacement, line 7 is justified by the commutativity of the partial differential operators, and in line 8 we replace dummy indices ($d \leftrightarrow i$ and $c \to j$) in the second term. As we see, the covariant derivative of $\mathbf{A}$ transforms as a tensor and hence it is a tensor. We can similarly show that the covariant derivative of a contravariant vector also transforms as a tensor and hence it is a tensor.

Now, the covariant derivative of a rank-0 tensor is the same as its partial derivative and hence it is a tensor (see exercises 2 and 3). We also demonstrated above that the covariant derivative of rank-1 tensors (i.e. vectors) is a tensor. As for tensors of ranks higher than 1, they are made of combinations of covariant and contravariant indices and hence we can simply generalize the pattern of the covariant and contravariant vectors; therefore their covariant derivative should transform as a tensor and hence it is a tensor.[71] We therefore conclude that the covariant derivative of a tensor of any rank and of any variance type is a tensor.

8. Show that the absolute derivative of a tensor is a tensor.

   Answer: The absolute derivative of a tensor along a curve is the inner product of the covariant derivative of the tensor and the tangent vector to the curve. Now, since both the covariant derivative of the tensor and the tangent vector to the curve are tensors (see exercise 7), and because the inner product of tensors is a tensor (see exercise 10 of § 2.2), then the absolute derivative is a tensor.

   Note: the validity of the above argument is based on the presumed existence of the covariant derivative of the tensor as a tensor field and this is not guaranteed because some tensors are defined only over a curve (e.g. momentum). However, this restriction on the validity of the argument may be lifted by the unique nature of the absolute derivative or by presumed extension of the tensor domain. Anyway, the above argument is useful from a pedagogical perspective. Moreover, there are more general (but rather lengthy) arguments (which are based on the employment of the basic transformation rules of tensors) that can be used for establishing the above premise.

9. Derive, from first principles, the mathematical expression for the intrinsic derivative of a covariant vector $\mathbf{A} = A_i\mathbf{E}^i$ with full justification of each step.

   Answer: The intrinsic derivative of a tensor is a total derivative of the tensor that includes differentiating the basis vectors as well as differentiating the components. Hence, we have:

$$\begin{aligned}
\frac{d\mathbf{A}}{dt} &= \frac{d}{dt}\left(A_i\mathbf{E}^i\right) \\
&= \mathbf{E}^i\frac{dA_i}{dt} + A_i\frac{d\mathbf{E}^i}{dt} \\
&= \mathbf{E}^i\frac{dA_i}{dt} + A_i\frac{\partial\mathbf{E}^i}{\partial x^j}\frac{dx^j}{dt} \\
&= \mathbf{E}^i\frac{dA_i}{dt} + A_i\left(-\Gamma^i_{kj}\mathbf{E}^k\right)\frac{dx^j}{dt} \\
&= \mathbf{E}^i\frac{dA_i}{dt} - \mathbf{E}^iA_k\Gamma^k_{ij}\frac{dx^j}{dt} \\
&= \left(\frac{dA_i}{dt} - A_k\Gamma^k_{ij}\frac{dx^j}{dt}\right)\mathbf{E}^i \\
&\equiv \frac{\delta A_i}{\delta t}\mathbf{E}^i
\end{aligned}$$

   where in line 1 we substitute for $\mathbf{A}$, in line 2 we use the product rule of differentiation, in line 3 we use the chain rule of differentiation, in line 4 we use the fundamental definition of Γ^i_{kj} (i.e. $\partial_j\mathbf{E}^i = -\Gamma^i_{kj}\mathbf{E}^k$), in line 5 we exchange the dummy indices i and k in the second term since they are just labels restricted to their term with no far reaching significance, in line 6 we take the common factor $\mathbf{E}^i$, and in line 7 we use the conventional notation of intrinsic derivative.

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[71] Tensors of higher ranks may also be seen in some cases as products of vectors and hence their covariant derivative should be a tensor. For example, if $A^i_j \equiv B^iC_j$ then $A^i_{j;k} = B^i_{;k}C_j + B^iC_{j;k}$ and hence $A^i_{j;k}$ is a tensor (because it is a sum of products of tensors; see exercises 7 and 8 of § 2.2) although this depends on the validity of the product rule independently.

10. Using the expressions for the intrinsic derivative of the tensors A_i and B^i along a curve $x^j(t)$, show that the intrinsic derivative is an inner product of the covariant derivative of the tensors and the tangent vector dx^j/dt to the curve.
Answer: We have:

$$\begin{aligned}\frac{\delta A_i}{\delta t} &= \frac{dA_i}{dt} - A_k \Gamma^k_{ij} \frac{dx^j}{dt} \\ &= \frac{\partial A_i}{\partial x^j}\frac{dx^j}{dt} - A_k \Gamma^k_{ij} \frac{dx^j}{dt} \\ &= \left(\frac{\partial A_i}{\partial x^j} - A_k \Gamma^k_{ij}\right) \frac{dx^j}{dt} \\ &= A_{i;j} \frac{dx^j}{dt}\end{aligned}$$

where in line 2 we use the chain rule, and the last line is the expression of the inner product of $A_{i;j}$ (which is the covariant derivative of A_i) and dx^j/dt (which is the tangent vector to the curve). Similarly:

$$\begin{aligned}\frac{\delta B^i}{\delta t} &= \frac{dB^i}{dt} + B^k \Gamma^i_{kj} \frac{dx^j}{dt} \\ &= \frac{\partial B^i}{\partial x^j}\frac{dx^j}{dt} + B^k \Gamma^i_{kj} \frac{dx^j}{dt} \\ &= \left(\frac{\partial B^i}{\partial x^j} + B^k \Gamma^i_{kj}\right) \frac{dx^j}{dt} \\ &= B^i_{;j} \frac{dx^j}{dt}\end{aligned}$$

where the last line is the expression of the inner product of $B^i_{;j}$ (which is the covariant derivative of B^i) and dx^j/dt (which is the tangent vector to the curve).
Note: the above demonstrations can be reversed and hence they can be used to show that from the characteristic of the intrinsic derivative as an inner product we obtain its formal expression.

11. Make a comparison between covariant derivative and absolute derivative.
Answer: We note the following:
• Covariant derivative is a generalization of ordinary partial derivative while absolute derivative is a generalization of ordinary total derivative (along a curve).
• Absolute derivative is defined along a curve in a space while covariant derivative is not restricted to a curve and hence it is generally defined over an entire space (or region of space).
• Absolute derivative includes (in a sense) covariant derivative since it is an inner product of the covariant derivative of a tensor and the tangent vector of a curve.
• The pattern of the two derivatives is very similar since both include an ordinary derivative term (i.e. total or partial derivative of tensor component) and a sum of positive Christoffel symbol terms (corresponding to contravariant indices) and negative Christoffel symbol terms (corresponding to covariant indices). This should be appreciated by noting that absolute derivative is an inner product of covariant derivative and a tangent vector (which does not disturb the essential pattern of the covariant derivative) and hence its pattern is essentially the same as the pattern of covariant derivative.

12. Show that the covariant derivative of the basis vectors vanishes identically.
Answer: We have:

$$\mathbf{E}_{i;j} = \partial_j \mathbf{E}_i - \Gamma^k_{ij}\mathbf{E}_k = +\Gamma^k_{ij}\mathbf{E}_k - \Gamma^k_{ij}\mathbf{E}_k = \mathbf{0}$$

$$\mathbf{E}^i_{;j} \quad = \quad \partial_j \mathbf{E}^i + \Gamma^i_{kj}\mathbf{E}^k = -\Gamma^i_{kj}\mathbf{E}^k + \Gamma^i_{kj}\mathbf{E}^k = \mathbf{0}$$

where in the first equalities we used the rules of covariant differentiation, and in the second equalities we used the fundamental definition of the Christoffel symbols, i.e. $\partial_j \mathbf{E}_i = \Gamma^k_{ij}\mathbf{E}_k$ and $\partial_j \mathbf{E}^i = -\Gamma^i_{kj}\mathbf{E}^k$.

13. Show that the covariant and intrinsic derivatives of the metric tensor and the Kronecker delta tensor vanish identically.

Answer: Regrading the covariant derivative we have:

$$\begin{aligned} g_{ij;k} &= \left(\mathbf{E}_i \cdot \mathbf{E}_j\right)_{;k} = \mathbf{E}_{i;k} \cdot \mathbf{E}_j + \mathbf{E}_i \cdot \mathbf{E}_{j;k} = 0 \\ g^{ij}_{\ \ ;k} &= \left(\mathbf{E}^i \cdot \mathbf{E}^j\right)_{;k} = \mathbf{E}^i_{;k} \cdot \mathbf{E}^j + \mathbf{E}^i \cdot \mathbf{E}^j_{;k} = 0 \\ g^j_{i;k} &= \delta^j_{i;k} = \left(\mathbf{E}_i \cdot \mathbf{E}^j\right)_{;k} = \mathbf{E}_{i;k} \cdot \mathbf{E}^j + \mathbf{E}_i \cdot \mathbf{E}^j_{;k} = 0 \end{aligned}$$

where we used the product rule (see exercise 18) as well as the result of the previous exercise.

Regrading the intrinsic derivative, it is an inner product of the covariant derivative and the tangent vector and hence it should also vanish identically (since the covariant derivative vanishes identically).

Note: the use of exercise 18 here could lead to circularity (because the proof of the application of the product rule to the inner product in that exercise rests on the results of the present exercise). So, to avoid circularity we should either use other methods for establishing the results of this exercise[72] (refer for example to B3 and B3X) or in exercise 18 we use a rather different argument for establishing the result of the inner product (i.e. by treating the inner product as an outer product between tensors which are already contracted, as explained in the note of exercise 18).

14. Elucidate the fact that tensor differential operators do not commute with each other in general.

Answer: It can be shown (see § 5.2 and § 7.2.1 of B3, and refer to exercises 5.52 and 7.16 of B3X) that for a covariant vector **A** and a contravariant vector **B** we have:

$$A_{j;kl} - A_{j;lk} = R^i_{\ jkl}A_i \qquad \text{and} \qquad B^j_{\ ;kl} - B^j_{\ ;lk} = R^j_{\ ilk}B^i$$

where the indexed R is the Riemann-Christoffel curvature tensor (see § 2.10). Now, since the Riemann-Christoffel curvature tensor vanishes identically *iff* the space is flat we conclude that tensor differential operators commute in flat spaces but not in curved spaces (and hence they do not commute in general).

Regrading the intrinsic derivative, it is an inner product of the covariant derivative and the tangent vector and hence it should also be non-commutative.

Note: $A_{j;kl}$ means $\left(A_{j;k}\right)_{;l}$ and $A_{j;lk}$ means $\left(A_{j;l}\right)_{;k}$.

15. Show that tensor differential operators commute with the index shifting operators and the index replacement operator.

Answer: This is a result of the "constancy" of the metric and Kronecker delta tensors with respect to tensor differentiation (which we established in exercise 13) plus the product rule (which we established in exercise 18). For example:

$$\begin{aligned} \left(g^{ij}A_j\right)_{;k} &= g^{ij}_{\ \ ;k}A_j + g^{ij}A_{j;k} = 0 + g^{ij}A_{j;k} = g^{ij}A_{j;k} = g^{ij}\left(A_{j;k}\right) \\ \left(\delta^j_i A_j\right)_{;k} &= \delta^j_{i;k}A_j + \delta^j_i A_{j;k} = 0 + \delta^j_i A_{j;k} = \delta^j_i A_{j;k} = \delta^j_i\left(A_{j;k}\right) \end{aligned}$$

The above should similarly apply to intrinsic derivative.

Note: the use of exercise 18 here may also lead to circularity (as in exercise 13 although it is less evident here). However, the use of the product rule here may be redundant if we use the "constancy" property plus the "linearity" property.[73] Alternatively, we may use contraction before and after covariant differentiation separately and compare the results, and hence the above examples become:

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[72] For example, we can use the rules of covariant differentiation, that is: $\delta^j_{i;k} = \partial_k \delta^j_i - \delta^j_l \Gamma^l_{ik} + \delta^l_i \Gamma^j_{lk} = 0 - \Gamma^j_{ik} + \Gamma^j_{ik} = 0$. Similarly, $g_{ij;k} = \partial_k g_{ij} - g_{lj}\Gamma^l_{ik} - g_{il}\Gamma^l_{jk} = \partial_k g_{ij} - [ik, j] - [jk, i] = 0$ where we used the result of exercise 5 of § 2.6. The relation $g^{ij}_{\ \ ;k} = 0$ can also be proved using the rules of covariant differentiation with similar relations and identities.

[73] The linearity property here means the derivative of a *constant times a function* is equal to the constant times the derivative of the function, e.g. $\frac{d(af)}{dt} = a\frac{df}{dt}$. However, this could also be circular.

$$\begin{aligned}\left(g^{ij}A_j\right)_{;k} &= \left(A^i\right)_{;k} = A^i_{;k} = g^{ij}\left(A_{j;k}\right)\\ \left(\delta^j_i A_j\right)_{;k} &= \left(A_i\right)_{;k} = A_{i;k} = \delta^j_i\left(A_{j;k}\right)\end{aligned}$$

In fact, these may be used to demonstrate the "constancy" of the metric and Kronecker delta tensors (with no need for the product rule).

16. Show that tensor differential operators commute with the contraction of indices.
Answer: If we note that the contraction of indices is conducted by the index shifting operators (e.g. $g_{ij}A^iB^j$ and $g^{ij}A_iB_j$) and the index replacement operator (e.g. $\delta^i_jA_iB^j$) then this is a result of the "constancy" of the metric and Kronecker delta tensors with respect to tensor differentiation (which we established in exercise 13). In other words, the result can be obtained from the result of exercise 15. For example:

$$\begin{aligned}\left(A^iB_i\right)_{;k} &= \left(g_{ij}A^iB^j\right)_{;k} = g_{ij}\left(A^iB^j\right)_{;k} = g_{ij}\left[\left(A^iB^j\right)_{;k}\right]\\ \left(A_iB^i\right)_{;k} &= \left(g^{ij}A_iB_j\right)_{;k} = g^{ij}\left(A_iB_j\right)_{;k} = g^{ij}\left[\left(A_iB_j\right)_{;k}\right]\\ \left(A_iB^i\right)_{;k} &= \left(\delta^i_jA_iB^j\right)_{;k} = \delta^i_j\left(A_iB^j\right)_{;k} = \delta^i_j\left[\left(A_iB^j\right)_{;k}\right]\end{aligned}$$

where on the far left we have covariant differentiation of contracted tensors while on the far right we have contraction of covariant derivatives of tensors.
The above should also apply to intrinsic derivative.

17. Show that the sum rule of differentiation applies to tensor differentiation like ordinary differentiation.
Answer: For example, if a and b are scalar constants and C^i_j and D^i_j are rank-2 mixed tensors then we have:

$$\begin{aligned}\left(aC^i_j \pm bD^i_j\right)_{;k} &= \left(aC^i_j \pm bD^i_j\right)_{,k} + \Gamma^i_{lk}\left(aC^l_j \pm bD^l_j\right) - \Gamma^l_{jk}\left(aC^i_l \pm bD^i_l\right)\\ &= \left(aC^i_{j,k} \pm bD^i_{j,k}\right) + \left(a\Gamma^i_{lk}C^l_j \pm b\Gamma^i_{lk}D^l_j\right) - \left(a\Gamma^l_{jk}C^i_l \pm b\Gamma^l_{jk}D^i_l\right)\\ &= \left(aC^i_{j,k} + a\Gamma^i_{lk}C^l_j - a\Gamma^l_{jk}C^i_l\right) \pm \left(bD^i_{j,k} + b\Gamma^i_{lk}D^l_j - b\Gamma^l_{jk}D^i_l\right)\\ &= a\left(C^i_{j,k} + \Gamma^i_{lk}C^l_j - \Gamma^l_{jk}C^i_l\right) \pm b\left(D^i_{j,k} + \Gamma^i_{lk}D^l_j - \Gamma^l_{jk}D^i_l\right)\\ &= aC^i_{j;k} \pm bD^i_{j;k} \qquad (10)\end{aligned}$$

where in line 1 and line 5 we use the rules of covariant differentiation and in line 2 we use the sum rule of partial derivative while the rest is based on simple algebraic manipulations. This can be easily generalized to any algebraic sums (or rather linear combinations) of tensors of any rank and of any variance type.
All this is about covariant differentiation. Regarding absolute differentiation, it is an inner product of covariant derivative and tangent vector (at least from the perspective of superficial pattern) and hence it should also follow the above pattern. For example:

$$\begin{aligned}\frac{\delta}{\delta t}\left(aC^i_j \pm bD^i_j\right) &= \left(aC^i_j \pm bD^i_j\right)_{;k}\frac{dx^k}{dt}\\ &= \left(aC^i_{j;k} \pm bD^i_{j;k}\right)\frac{dx^k}{dt}\\ &= aC^i_{j;k}\frac{dx^k}{dt} \pm bD^i_{j;k}\frac{dx^k}{dt}\\ &= a\frac{\delta}{\delta t}C^i_j \pm b\frac{\delta}{\delta t}D^i_j\end{aligned}$$

where in line 2 we use Eq. 10. Therefore, we conclude that the sum rule (or rather the linearity) of ordinary differentiation also applies to tensor differentiation.

18. Show that the product rule of differentiation applies to tensor differentiation like ordinary differentiation.
Answer: For example, if B^iC_j is a product of two tensors then we have:

$$\begin{aligned}\left(B^iC_j\right)_{;k} &= \left(B^iC_j\right)_{,k} + \Gamma^i_{lk}\left(B^lC_j\right) - \Gamma^l_{jk}\left(B^iC_l\right)\\ &= B^i_{,k}C_j + B^iC_{j,k} + \Gamma^i_{lk}B^lC_j - \Gamma^l_{jk}B^iC_l\\ &= \left(B^i_{,k} + \Gamma^i_{lk}B^l\right)C_j + B^i\left(C_{j,k} - \Gamma^l_{jk}C_l\right)\\ &= B^i_{;k}C_j + B^iC_{j;k} \qquad (11)\end{aligned}$$

where in line 1 and line 4 we use the rules of covariant differentiation. As we see, the last line is the product rule (applied to covariant differentiation). This can be easily generalized to any products of tensors of any rank and of any variance type.
All this is about covariant differentiation. Regarding absolute differentiation, it is an inner product of covariant derivative and tangent vector (at least superficially) and hence it should also follow the above pattern. For example:

$$\begin{aligned}\frac{\delta}{\delta t}\left(B^iC_j\right) &= \left(B^iC_j\right)_{;k}\frac{dx^k}{dt}\\ &= \left(B^i_{;k}C_j + B^iC_{j;k}\right)\frac{dx^k}{dt}\\ &= B^i_{;k}\frac{dx^k}{dt}C_j + B^iC_{j;k}\frac{dx^k}{dt}\\ &= \frac{\delta B^i}{\delta t}C_j + B^i\frac{\delta C_j}{\delta t}\end{aligned}$$

where in line 2 we use Eq. 11. As we see, the last line is the product rule (applied to absolute differentiation). This can be easily generalized to any products of tensors of any rank and of any variance type. Therefore, we conclude that the product rule of ordinary differentiation also applies to tensor differentiation (i.e. covariant and absolute).
All this is about the application of the product rule to the outer product. Regarding the inner product, it is an outer product followed by a contraction of indices and hence from the above results plus the result of exercise 16 the product rule should also apply. For example, $B^iC_i = \delta^j_iB^iC_j$ and hence if we use the above results then we have:

$$\begin{aligned}\left(B^iC_i\right)_{;k} &= \left(\delta^j_iB^iC_j\right)_{;k} = \delta^j_i\left(B^iC_j\right)_{;k} = \delta^j_i\left(B^i_{;k}C_j + B^iC_{j;k}\right) = B^i_{;k}C_i + B^iC_{i;k}\\ \frac{\delta}{\delta t}\left(B^iC_i\right) &= \frac{\delta}{\delta t}\left(\delta^j_iB^iC_j\right) = \delta^j_i\frac{\delta}{\delta t}\left(B^iC_j\right) = \delta^j_i\left(\frac{\delta B^i}{\delta t}C_j + B^i\frac{\delta C_j}{\delta t}\right) = \frac{\delta B^i}{\delta t}C_i + B^i\frac{\delta C_i}{\delta t}\end{aligned}$$

Note: we may treat the inner product as an outer product between tensors which are already contracted[74] and hence the use of the result of exercise 16 (which is based on the "constancy" of the contraction operator) will become redundant. Accordingly, the application of the product rule to inner product should be an instance (or special case) of the application of the product rule in outer product, and hence the above examples will become as follows:

$$\begin{aligned}\left(\delta^j_iB^iC_j\right)_{;k} &= \left(B^iC_i\right)_{;k} = B^i_{;k}C_i + B^iC_{i;k}\\ \frac{\delta}{\delta t}\left(\delta^j_iB^iC_j\right) &= \frac{\delta}{\delta t}\left(B^iC_i\right) = \frac{\delta B^i}{\delta t}C_i + B^i\frac{\delta C_i}{\delta t}\end{aligned}$$

19. What is the link between the following premises: "*the intrinsic derivative is a total derivative of the tensor that includes differentiating the basis vectors*" and "*the intrinsic derivative is the inner product*

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[74] Although this may be difficult to justify conceptually, it should be fine formally.

of the covariant derivative of the tensor and the tangent vector"?
Answer: The link is the chain rule, that is:

$$\frac{\delta \mathbf{A}}{\delta t} \equiv \frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial x^k}\frac{dx^k}{dt} \equiv \mathbf{A}_{;k}\frac{dx^k}{dt}$$

where the derivatives $d\mathbf{A}/dt$ and $\partial\mathbf{A}/\partial x^k$ include differentiating the basis vectors of the tensor $\mathbf{A}$ (noting that $\frac{\delta \mathbf{A}}{\delta t}$ and $\mathbf{A}_{;k}$ represent an abuse of notation).

20. The rank and variance type of the intrinsic derivative is the same as the rank and variance type of the differentiated tensor. Why?
Answer: The intrinsic derivative may be seen as the inner product of the covariant derivative of the tensor and the tangent vector to the curve. This means that the covariant rank of the tensor increases by 1 (from covariant differentiation) and its contravariant rank also increases by 1 (from the outer product with the tangent vector since inner product includes outer product). However, these added ranks are consumed by the contraction (which is embedded in the inner product) and hence the rank and variance type of the differentiated tensor remain the same. In other words, although we add a covariant index by covariant differentiation and a contravariant index by outer multiplication with the tangent vector, we contract these indices (which means they disappear as free indices) and hence the rank and variance type of the original tensor remain unaffected.

2.8 Parallel Transport

1. Give a brief definition of parallel transport considering both flat space and curved space.
Answer: Parallel transport of a vector means moving the vector from one position in the space to another position in the space while keeping its magnitude and direction constant. In flat space this is obvious because direction is independent of position and hence parallel transport is independent of the path of transport. However, in curved space direction is dependent on position, due to the intrinsic curvature of the space, and hence parallel transport can only be defined in reference to a given curve in the space.
Note: in fact, the independence in flat space and the dependence in curved space generally includes even magnitude and is not restricted to direction because in curved space even the magnitude generally depends on the local metric properties and hence on the position, i.e. the magnitude is generally local and not global (although this may depend on the nature of the parallel-transported vector). However, in the context of parallel transport we generally restrict our attention to direction because it is the most distinctive feature of parallelism (due to the intuitive association between parallelism and direction) and because of the potential independence of the magnitude of some vector quantities from position even in curved space. Moreover, direction is sufficient to make the distinction between flat space and curved space. This should also ease the presentation and reduce the risk of confusion.
2. Explain how the definition of parallel transport of a vector along a curve is formally implemented.
Answer: As explained earlier, parallel transport means moving a vector along a curve while keeping it constant in magnitude and direction intrinsically. This is formally implemented by setting the absolute derivative of the vector along the curve identically zero because the absolute derivative (which is also known as *intrinsic* derivative) quantifies the change of the vector along the curve intrinsically. So, if we want to keep the vector constant intrinsically along the curve then we should make its variation (which is measured by the absolute derivative) along the curve zero.
3. Justify the fact that parallel transport is path dependent.
Answer: This is because in curved space "direction" (which the concept of parallel transport rests upon) is position dependent and hence the path (which is a continuous series of positions) should determine the direction of the parallel-transported vector at each position (including the final position). In other words, parallel transport is path dependent. Also see the note of exercise 1.
4. Justify the fact that starting from a given point P on a closed curve C, parallel-transporting a vector field around C does not necessarily result in the same vector field when arriving back at P.
Answer: This is justified by the previous exercise because a closed path is made of two sub-paths

connecting P to another point (where one of these two sub-paths is reversed). Hence, intuitively the result of parallel transport around a closed path should depend on the closed path (which is made of the two sub-paths) and therefore the final vector is not necessarily the same as the initial vector when the vector field returns to its starting point.
To be more clear, let have a closed path C that contains two distinct points, P_1 and P_2, and hence C is made of two sub-paths: C_1 which goes from P_1 to P_2, and C_2 which goes from P_2 to P_1 (see Figure 2). Also, let have a vector $\mathbf{v}_1$ at P_1 that we want to parallel-transport around C. Now, since the vector (say $\mathbf{v}_2$) obtained at P_2 by parallel-transporting $\mathbf{v}_1$ along C_1 is dependent on C_1 and the vector (say $\mathbf{v}_3$) obtained at P_1 by parallel-transporting $\mathbf{v}_2$ along C_2 is dependent on C_2 then $\mathbf{v}_3$ should depend on C (since C is made of C_1 and C_2) and hence $\mathbf{v}_3$ is not necessarily identical to $\mathbf{v}_1$ (or rather $\mathbf{v}_3$ is generally different from $\mathbf{v}_1$). The situation becomes more clear if we compare the vector obtained by parallel-transporting $\mathbf{v}_2$ backward along C_1 (where it becomes $\mathbf{v}_1$ at P_1) with the vector obtained by parallel-transporting $\mathbf{v}_2$ along C_2 (where it becomes $\mathbf{v}_3$ at P_1) because due to the path dependency $\mathbf{v}_1$ and $\mathbf{v}_3$ should be different in general due to the dependency of the parallel transport of $\mathbf{v}_2$ on C_1 and C_2. In fact, we can change the argument by starting from $\mathbf{v}_2$ where it becomes (by parallel transport along C_1) $\mathbf{v}_1$ and it becomes (by parallel transport along C_2) $\mathbf{v}_3$ where $\mathbf{v}_1$ and $\mathbf{v}_3$ are different in general due to the path dependency, and hence the vector obtained by parallel-transporting along the loop starting from P_1 (whether this vector is $\mathbf{v}_1$ or $\mathbf{v}_3$) should be different when it returns to P_1 (whether it returns as $\mathbf{v}_3$ or $\mathbf{v}_1$).
Note: the situation in parallel transport is similar to the situation in non-conservative vector fields and the argument for their path dependency which leads to the conclusion that the work done along a closed loop is not zero. In brief, parallel-transporting of vectors around a loop in flat space is like transporting objects around a loop in conservative fields while parallel-transporting of vectors around a loop in curved space is like transporting objects around a loop in non-conservative fields.

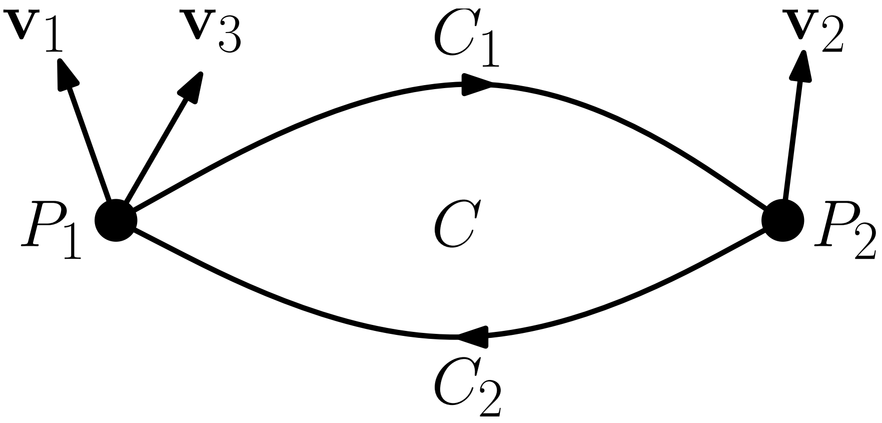

Figure 2: Parallel transport of a vector $\mathbf{v}_1$ from point P_1 to point P_2 along curve C_1 (where it becomes $\mathbf{v}_2$) then back to P_1 along curve C_2 (where it becomes $\mathbf{v}_3$).

5. Justify, qualitatively, the fact that the unit tangent vector of a geodesic curve is a parallel-transported field along the curve.
   Answer: The unit tangent vector will keep its magnitude (since it is a unit vector) and direction (since geodesic curve follows the intrinsic curvature of the space with no deviation; see § 2.9.1) and hence it satisfies the definition of parallel transport (i.e. transporting a vector along a curve while keeping its magnitude and direction constant intrinsically). This may also be justified further by the fact that geodesics in curved space are the equivalent to straight lines in flat space. More details about these issues will be given later.
6. Show, formally, that the magnitude of a parallel-transported vector field along a curve is constant.
   Answer: The magnitude $|\mathbf{v}|$ of a vector $\mathbf{v}$ is given by $|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v}$. If $\mathbf{v}$ is a parallel-transported vector

field along a t-parameterized curve then we have (assuming $|\mathbf{v}| \neq 0$):

$$\begin{aligned}
\frac{d|\mathbf{v}|^2}{dt} &= \frac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) \\
2|\mathbf{v}|\frac{d|\mathbf{v}|}{dt} &= \frac{d\mathbf{v}}{dt}\cdot\mathbf{v}+\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} \\
2|\mathbf{v}|\frac{d|\mathbf{v}|}{dt} &= 2\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} \\
|\mathbf{v}|\frac{d|\mathbf{v}|}{dt} &= \mathbf{v}\cdot\frac{d\mathbf{v}}{dt} \\
|\mathbf{v}|\frac{d|\mathbf{v}|}{dt} &= \mathbf{v}\cdot\frac{\delta\mathbf{v}}{\delta t}
\end{aligned}$$

where in line 2 we use the power, chain and product rules of differentiation, in line 3 we use the commutativity of inner product, and in line 5 we use the fact that in general coordinate systems the total derivative is equivalent to absolute derivative (see § 2.7).[75] Now, the absolute derivative of a parallel-transported vector is zero (i.e. $\frac{\delta\mathbf{v}}{\delta t}=\mathbf{0}$) and $|\mathbf{v}|\neq 0$. Hence, $\frac{d|\mathbf{v}|}{dt}=0$ which means that the magnitude of the parallel-transported vector field is constant along the curve.

7. Show that the angle between two parallel-transported vector fields along a given curve is constant.
Answer: The angle θ between two vectors, $\mathbf{v}$ and $\mathbf{w}$, is given by (assuming $|\mathbf{v}|\neq 0$ and $|\mathbf{w}|\neq 0$):

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}|\,|\mathbf{w}|}$$

If $\mathbf{v}$ and $\mathbf{w}$ are parallel-transported vector fields along a given t-parameterized curve then we have:

$$\begin{aligned}
\frac{d}{dt}(\cos\theta) &= \frac{d}{dt}\left(\frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}|\,|\mathbf{w}|}\right) \\
&= \frac{1}{|\mathbf{v}|\,|\mathbf{w}|}\frac{d}{dt}(\mathbf{v}\cdot\mathbf{w}) \\
&= \frac{1}{|\mathbf{v}|\,|\mathbf{w}|}\left(\frac{d\mathbf{v}}{dt}\cdot\mathbf{w}+\mathbf{v}\cdot\frac{d\mathbf{w}}{dt}\right) \\
&= \frac{1}{|\mathbf{v}|\,|\mathbf{w}|}\left(\frac{\delta\mathbf{v}}{\delta t}\cdot\mathbf{w}+\mathbf{v}\cdot\frac{\delta\mathbf{w}}{\delta t}\right) \\
\frac{d}{dt}(\cos\theta) &= 0
\end{aligned}$$

where line 2 is because $|\mathbf{v}|$ and $|\mathbf{w}|$ are constants according to the result of the previous exercise, line 3 is the product rule, line 4 is because in general coordinate systems the total derivative is equivalent to absolute derivative (see § 2.7), and line 5 is because the absolute derivative of parallel-transported vector is zero (i.e. $\frac{\delta\mathbf{v}}{\delta t}=\frac{\delta\mathbf{w}}{\delta t}=\mathbf{0}$). Hence, $\cos\theta$ is constant which implies that θ is constant.

8. What you conclude from the last two exercises?
Answer: The results of the last two exercises demonstrate the fact that in parallel transport the parallel-transported vector keeps its magnitude and direction (in intrinsic sense) and hence it is constant in this sense.

9. Show that the inner product of two parallel-transported vectors along a given curve is constant.
Answer: This can be seen from the definition of inner product (i.e. $\mathbf{v}\cdot\mathbf{w}=|\mathbf{v}|\,|\mathbf{w}|\cos\theta$) because $|\mathbf{v}|$, $|\mathbf{w}|$ and $\cos\theta$ are constants according to the results of exercises 6 and 7.

10. Justify the fact that parallel transport is field dependent.[76]
Answer: This can be concluded from the results of the previous exercises since each one of the two

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[75] We note that the absolute derivative of scalar is the same as its total derivative (and hence we did not change $\frac{d|\mathbf{v}|}{dt}$).

[76] This may sound trivial but the purpose of it is that two distinct vector fields that are parallel-transported along a given path will remain distinct at each point of the path.

parallel-transported vector fields (which are assumed different) will keep its magnitude and its direction relative to the other field and hence they will be different at each point on the curve. In fact, the result of exercise 7 is more specific about this issue since the constancy of the angle θ along the path will keep the two vectors distinct at all points of the path.

11. Justify the fact that parallel transport provides an indicator of the curvature of space (i.e. in flat space a parallel-transported vector around a closed path returns to its starting point in the same direction as that at the starting point, while in curved space it generally returns to its starting point in a different direction to that at the starting point).
Answer: This is justified by the fact that in flat space direction is independent of position (and hence independent of the path of transport) because direction has a global sense. So, if we move a vector while keeping its direction parallel to a global direction then it will necessarily return to its initial position in the same direction because in each step along the path we are referring the vector to the same global and unique direction. On the other hand, in curved space direction is dependent on position (and hence dependent on the path of transport) because direction has only local sense. So, if we move a vector while keeping its direction parallel to a local direction (i.e. its direction in the start of an infinitesimal step) then it will not necessarily return to its initial position in the same direction because in each step along the path we are referring the vector to a position-dependent local direction and therefore the final direction will depend on the path of transport. In other words, the reference direction[77] can be gradually modified in each step along the path (due to the dependence of direction on position in curved space) and hence the vector will not return back in the same direction as its initial direction at the starting point.
12. Although parallel transport is primarily defined in relation to vectors, it can be extended to higher rank tensors. How?
Answer: The extension is based on applying similar conditions and criteria as those for vectors; the main of these criteria is the vanishing of the absolute derivative of the tensor along the curve. For example, if $\mathbf{A}$ is a rank-n tensor that is parallel-transported along a t-parameterized curve then we should have $\frac{\delta \mathbf{A}}{\delta t} = \mathbf{0}$ identically along the entire curve.

2.9 Geodesic Path

1. Discuss the idea of being optimal or extremal (as a criterion for geodesics) in the relativity theories.
Answer: As we will see, the idea of being optimal or extremal is redundant (or irrelevant) in the relativity theories although it may be used as a starting point and motivation for the definition of geodesic. This can be appreciated by considering the planetary orbits (as well as many other examples of free motion of massive and massless objects in spacetime) as geodesics. So, the essential criterion for geodesics in the relativity theories is the fulfillment of the condition set by the geodesic equation (see § 2.9.5).
2. Compare between timelike geodesics and null geodesics within the framework of relativity theories.
Answer: We note the following:
• Both are geodesics in technical geometric sense (according to the condition of the geodesic equation).
• Timelike geodesics are the paths of free massive objects in spacetime, while null geodesics are the paths of free massless objects in spacetime.[78]
• For timelike geodesics $(d\sigma)^2 > 0$ (i.e. timelike spacetime interval) while for null geodesics $(d\sigma)^2 = 0$ (i.e. lightlike spacetime interval) where $(d\sigma)^2$ is the quadratic form of the spacetime.
3. What is the difference between special relativity and general relativity with regard to geodesics and null geodesics?

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[77] The reference direction may be identified by saying "the vector is kept parallel to itself" in a sense that in each infinitesimal step along the path the vector is kept constant with no variation in magnitude or direction. This is equivalent to having vanishing rate of change which is our formal criterion for defining parallel transport.

[78] "Free" means they are not influenced by any force (noting that gravity in general relativity is not a force in a technical sense but an effect of the spacetime curvature). This may also be labeled in general relativity with terms like "free fall gravitation" which means motion in the absence of any force other than gravity.

Answer: The spacetime of special relativity is flat and hence geodesics and null geodesics are straight lines, while the spacetime of general relativity is curved and hence geodesics and null geodesics are twisted curves in general.
Note: intrinsically (i.e. as seen by an intrinsic inhabitant), there is no difference between straight geodesic and curved geodesic. Therefore, this distinction should be based either on extrinsic perspective (assuming the existence of embedding space) or on the nature of the spacetime (i.e. being flat or curved) or the patch of spacetime in which the geodesic exists.

4. Give a brief characterization of geodesic path in Euclidean and Riemannian spaces.
Answer: Geodesic path is a straight line in Euclidean space, but it is generally a curve in Riemannian space.
5. Why is the concept of gravitational force seen redundant in general relativity? What is the implication of this? How can this explain the correspondence between special relativity and general relativity?
Answer: The concept of force is seen redundant because the gravitationally-driven motion in the spacetime of general relativity is viewed as a natural tendency of (free) physical objects to move along the geodesic paths of the spacetime (i.e. the objects naturally follow geodesic trajectories) and hence the concept of geodesic is sufficient to explain the motion of objects with no need for a gravitational driving force.[79]
The obvious implication is that in general relativity the objects (whether massive or massless) that are under the influence of gravitational effects only are seen as free particles because they are free of the effect of any force.
The correspondence is obvious because although we are introducing gravity to the spacetime in general relativity the behavior of physical objects in the modified spacetime (assuming the absence of other forces) is essentially the same as that in special relativity since the objects in both spacetimes are free and they equally follow geodesic trajectories although the nature of these geodesic trajectories is different (i.e. straight in special relativity and curved in general relativity) due to the difference in the nature of these spacetimes (i.e. flat in special relativity and curved in general relativity). In fact, this in essence reflects the spirit of the equivalence principle (see § 1.8.2).
6. Compare between classical gravity and general relativistic gravity considering the concepts of mass and geodesic paths.
Answer: We note the following:
• In classical gravity we have gravitating mass and gravitated mass and the gravity is seen as an interaction (or force) between these two types of mass with no need for the concept of geodesic paths. In general relativistic gravity we have no gravitated mass, moreover the role of gravitating mass is modified from being a gravitating mass to be a source and cause for the geometric formation of spacetime and hence gravity is seen as a geometric effect by following geodesic paths in this formed geometry.
• The role of "gravitating mass" and "gravitated mass" in classical gravity is played equally by both masses that interact gravitationally (although we usually label the more massive as gravitating and the less massive as gravitated). This means that if M_A and M_B are interacting gravitationally then from one perspective M_A is gravitating and M_B is gravitated and from the opposite perspective M_B is gravitating and M_A is gravitated. In fact, this is a healthy relation since it reflects the symmetry of the physical situation and it is inline with the Newton's third law. Moreover, it applies intuitively in any situation regardless of the relative size of the two masses and to which the generated gravitational field is largely attributed. For example, when the two masses are equal we have completely symmetrical situation and hence labeling one as gravitating and the other as gravitated is entirely arbitrary and a matter of choice and convention since they attract each other equally. Moreover, their contribution to the generated gravitational field is equal. On the other hand, the relativistic paradigm of geodesic discriminates between the two masses since it assumes that one is generating the "gravitational field" (as represented by the geometry of spacetime) while the other is subjected to this "gravitational field" and hence it follows certain routes determined by this field. This does not reflect the actual physical situation especially when the gravitationally-interacting masses are of equal or comparable size

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[79] From this perspective, general relativity is similar to the ancient theories of "gravity" (e.g. Aristotelian) as explained in § 1.2.

since the generated "gravitational field" belongs to both (i.e. it is a superposition of their individual contributions) and hence we do not have a gravitated object that follows the geodesic trajectories of the "field" generated by the gravitating object. In other words, the paradigm of geodesic may apply aptly in the case of gravitational interaction between a (gravitated) test particle and a (gravitating) source of gravity but it does not apply in general where the gravitationally-interacting objects are of comparable size and hence both objects are sources of gravity.[80] The impact of this lack of symmetry is not only conceptual but even practical, as indicated earlier. This issue will be investigated further in the future (see for example § 10.1.18).

7. Can the paradigm of "geodesic" replace the paradigm of "gravitational force" entirely?
Answer: No. For example:
• The lack of symmetry in the geodesic paradigm (as explained in the previous exercise) dictates the need for the paradigm of force (which can apply symmetrically) in certain theoretical and practical contexts.
• The concept of "gravitational force" (although fictitious) may also be needed (according to some opinions) in a frame located in a gravitational field but the frame is not in a state of free fall relative to the source of gravity (i.e. frame of type c in the investigation of § 6.1). However, this may be refuted by the fictitious nature of the force. Moreover, it may be interpreted as an effect of acceleration according to the equivalence principle. Nevertheless, the insufficiency of the paradigm of "geodesic" to replace the paradigm of "gravitational force" is obvious in this case even if we take account of these challenges.
• We may need to borrow the paradigm of "gravitational force gradient" and the paradigm of "tidal force" from classical gravity since these paradigms do not exist in general relativity (thanks to the paradigm of geodesic and the equivalence principle). We note that the "geodesic deviation" of general relativity does not substitute the aforementioned classical paradigms entirely because the dynamical aspect of force is lost in the paradigm of geodesic deviation (see § 7.13 and § 10.1.18). We also question the global nature of geodesic deviation unlike these classical paradigms.
8. Is the paradigm of geodesic in general relativity a novelty of this theory?
Answer: Not really. In fact, the novelty of this paradigm in general relativity is highly questionable since this paradigm is used in classical physical theories related to the calculus of variations and the Lagrange-Hamilton mechanics. Yes, this paradigm may have some minor novel elements in the field of gravitation.
Note: those who are familiar with the Lagrangian formulation of Newton's second law, i.e.

$$m\left(\Gamma^i_{jk}\dot{q}^j\dot{q}^k + \ddot{q}^i\right) = -g^{ij}\partial_j V$$

should appreciate the presence of the "geodesic equation" within this formulation when "gravity" ceases to be a "force" (by embedding it in the geometry of spacetime) and hence the "force" source term vanishes. In fact, they can even see the equivalence principle by noting that $\Gamma^i_{jk}\dot{q}^j\dot{q}^k + \ddot{q}^i$ is the "acceleration" that corresponds to this embedded form of "gravity". In other words, the geodesic equation is no more than a declaration of the vanishing of "acceleration" in the spacetime when gravity is considered an attribute of the spacetime geometry and not a force (inline with Newton's first law which is a special case of Newton's second law corresponding to the absence of force).

2.9.1 Geodesic Path in 2D Spaces

1. Investigate potential difficulties in the definitions of natural and affine parameters as given in the above Problem especially in the context of general relativity.
Answer: In pseudo-Riemannian spaces the quadratic form is not positive definite and therefore the line element (and hence the "curve length") could be zero or imaginary.
With regard to the use of natural and affine parameters in the spacetime of general relativity (which is a pseudo-Riemannian space) the aforementioned definitions are problematic in the case of lightlike trajectories because the spacetime interval (which represents the "curve length") is null and therefore

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[80] We note that "both objects" and its alike represent the simplest case since more than two objects can participate in gravitational interaction. The generalization of the above arguments to these cases is straightforward.

the natural and affine parameters according to the above definitions are useless. To overcome this problem, the affine parameter λ in this case may be defined by the equation $x^\mu = \frac{dx^\mu}{d\lambda}\lambda$ where x^μ represents the coordinates of the trajectory of light.[81] Anyway, there are several ways for defining natural and affine parameters sensibly in this case. Moreover, our main use of the above definitions is in the case of timelike trajectories where we have no problem because in this case the spacetime interval (and hence the "curve length" in the above definition) is positive.
Regarding spacelike trajectories (where the line element is imaginary) the natural and affine parameters can be defined (if they are needed at all) by taking the magnitude of the line element.

2. Show that the magnitude of the tangent vector of a curve is constant when the curve is parameterized with natural or affine parameter.
Answer: Let the curve be represented by $\mathbf{r}(\lambda)$ where λ represents a natural or affine parameter.
When the curve is parameterized by natural parameter s then $\lambda \equiv s$ and hence the magnitude of its tangent is given by:
$$\left|\frac{d\mathbf{r}}{d\lambda}\right| = \left|\frac{d\mathbf{r}}{ds}\right| = \left|\frac{\sqrt{d\mathbf{r}\cdot d\mathbf{r}}}{ds}\right| = \left|\frac{ds}{ds}\right| = 1$$
When the curve is parameterized by affine parameter then $\lambda \equiv as$ (with a being a non-zero real constant) and hence the magnitude of its tangent is given by:
$$\left|\frac{d\mathbf{r}}{d\lambda}\right| = \left|\frac{d\mathbf{r}}{ds}\frac{ds}{d\lambda}\right| = \left|\frac{d\mathbf{r}}{ds}\right|\left|\frac{ds}{d\lambda}\right| = \left|\frac{ds}{d\lambda}\right| = \left|\frac{ds}{d\,(as)}\right| = \frac{1}{|a|}\left|\frac{ds}{ds}\right| = \frac{1}{|a|}$$
Hence, in both cases the magnitude of the tangent vector is constant.

2.9.2 Geodesic Equation from Straightness

1. Write the geodesic equation in a compact tensorial form.
Answer: We may write it as:
$$T^i_{;j}T^j = 0$$
where the indexed T is the unit tangent vector to the geodesic curve. This form can be justified by the fact that the absolute derivative of the unit tangent vector to the geodesic curve is zero plus the fact that the absolute derivative is an inner product of the covariant derivative $T^i_{;j}$ and the tangent vector T^j (where the unity of the tangent vector is justified by parameterizing the curve naturally since $T^a = dx^a/ds$).

2.9.3 Geodesic Equation from Variational Principle

1. State the mathematical condition for the variational principle in the spacetime of general relativity and interpret this condition in the case of timelike trajectory.
Answer: In the spacetime of general relativity s is replaced by σ and hence the variational principle becomes $\delta\int d\sigma = 0$. Now, if we consider a timelike trajectory and divide both sides by c we get $\delta\int d\tau = 0$ which means that the geodesic path optimizes the proper time τ for timelike trajectories of massive free objects.
2. Justify the following relation: $\partial_k \dot{x}^i = 0$ (as well as $\partial_k \dot{x}^j = 0$).
Answer: Since λ is independent of x^k then we have:
$$\partial_k \dot{x}^i \equiv \frac{\partial}{\partial x^k}\left(\frac{dx^i}{d\lambda}\right) = \frac{d}{d\lambda}\left(\frac{\partial x^i}{\partial x^k}\right) = \frac{d}{d\lambda}\delta^i_k = 0$$

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[81] Qualitatively, the affine parameter of a null geodesic may be defined as a parameter that is proportional to an affine parameter of a neighboring non-null geodesic.

3. Justify the claim that g_{ij} is independent of $\dot{x}^k$.
   Answer: It should be obvious that although g_{ij} is dependent on the coordinates x^k it cannot be dependent on the derivative of the coordinates with respect to the parameter λ of an arbitrary curve. In other words, g_{ij} is dependent on the location (or coordinates of a point in the space) but it is not dependent on the tangential direction of an arbitrary curve passing through that point because this violates the sole dependency of the metric on the basis vectors (according to the definition of metric) since the tangential direction of an arbitrary curve depends on extra parameters other than the basis vectors.
4. Justify the following relation: $\partial\dot{x}^i/\partial\dot{x}^k = \delta^i_k$.
   Answer: We have two cases:
   (**a**) $i = k$: in this case $\partial\dot{x}^i/\partial\dot{x}^k = \partial\dot{x}^i/\partial\dot{x}^i = 1$.
   (**b**) $i \neq k$: in this case x^i and x^k are independent of each other (since the coordinates are mutually independent) and hence their total derivatives (i.e. $\dot{x}^i$ and $\dot{x}^k$) with respect to a third parameter (i.e. λ) that is independent of x^i and x^k should also be independent of each other. The reason is that since λ is independent of x^i and x^k then the differentiation of x^i and x^k with respect to λ will not introduce any dependency of the derivatives on x^i or x^k. Therefore, $\partial\dot{x}^i/\partial\dot{x}^k = 0$.
   On combining these two cases we obtain $\partial\dot{x}^i/\partial\dot{x}^k = \delta^i_k$.

2.9.4 Geodesic Equation from Parallel Transport

1. State the mathematical condition for a vector field A^i to be parallel-transported along a given curve.
   Answer: A vector field A^i is parallel-transported along a given t-parameterized curve $x^j = x^j(t)$ *iff* its absolute derivative along the curve vanishes identically, that is:

$$\frac{\delta A^i}{\delta t} \equiv A^i_{\ ;j}\frac{dx^j}{dt} \equiv \frac{dA^i}{dt} + \Gamma^i_{kj}A^k\frac{dx^j}{dt} = 0$$

2. Briefly discuss the qualification of geodesic curve as "parallel-transporting its own tangent".
   Answer: A sufficient and necessary condition for a curve to be geodesic is that it is a tangent to a parallel vector field. This means that the tangent to a geodesic curve at a given point on the curve is characterized by keeping its tangential direction and hence it remains tangent to the curve as it moves (i.e. by parallel transport) along the curve. Accordingly, a vector attained by parallel transport of a tangent vector to a geodesic curve at a given point on the curve stays always tangent to the geodesic curve. As a result, a vector field attained by parallel transport along a geodesic makes a constant angle with the geodesic.
3. Show, qualitatively, that the geodesic criterion of "most straight" and the geodesic criterion of "parallel-transporting its tangent" are equivalent.
   Answer: The characteristic feature of being straight is that its tangent is constant (which implies parallel-transporting its tangent) and this characteristic feature is embedded in the criterion of "parallel-transporting its tangent" since the tangential direction will be constant along the curve according to the definition of parallel-transporting a vector field (i.e. keeping it constant along the path of transport).
4. Justify the fact that in parallel transport of the tangent vector along a geodesic curve the parameter of the curve should be natural (or affine).
   Answer: In fact, the condition of being natural (or affine) is needed to obtain the required mathematical condition (i.e. the stated form of the geodesic equation) and is not a fundamental requirement of parallel transport itself. The justification is that the tangent vector of a curve parameterized by a general parameter does not necessarily have constant magnitude.[82] However, in parallel transport the vector should keep its magnitude (as well as its direction) since the vector must remain constant. So, if we parameterize the curve with a natural (or affine) parameter then the tangent vector (as a

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[82] We consider in this statement the fact that the tangent vector $\mathbf{t}$ of an α-parameterized curve is the total derivative of the spatial representation $\mathbf{r}(\alpha)$ of the curve with respect to its parameter α, i.e. $\mathbf{t} = \frac{d\mathbf{r}}{d\alpha}$.

derivative with respect to this parameter) is constant in magnitude (which in the case of natural parameterization is unity). In this case, the parameter in the equation of the tangent to the curve (e.g. $\mathbf{t} = \frac{d\mathbf{r}}{d\alpha}$) can be unified with the parameter in the equation of parallel transport (which expresses the identical vanishing of the absolute derivative of the tangent) and hence the geodesic equation can be obtained in its stated form.
Note: we should remark that in the definition of the tangent vector we use ordinary total derivative (e.g. $\mathbf{t} = \frac{d\mathbf{r}}{d\alpha}$) rather than absolute derivative because the tangent vector belongs to the tangent space which is Euclidean (or flat). The reader should be aware that the tangent space at any point of a Riemannian (or pseudo-Riemannian) space is a Euclidean (or flat) space of the same dimensionality that represents the local geometry of the space. The existence of the tangent space is guaranteed by the Riemannian nature of the space since Riemannian space is locally flat.

5. Show that the geodesic equation can be written as $\frac{dT_i}{d\lambda} = \frac{1}{2}\partial_i g_{jk} T^j T^k$ (where the indexed T is the unit tangent vector and λ is a natural parameter).
Answer: If we use the covariant form of the unit tangent vector T_i then Eq. 70 (in the text) becomes $\frac{dT_i}{d\lambda} - T_k \Gamma^k_{ij} \frac{dx^j}{d\lambda} = 0$. Hence:

$$\begin{aligned}\frac{dT_i}{d\lambda} &= T_k \Gamma^k_{ij} \frac{dx^j}{d\lambda} \\ &= \Gamma^k_{ij} T_k T^j \\ &= \frac{g^{kl}}{2} \left(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij}\right) T_k T^j \\ &= \frac{1}{2} \left(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij}\right) T^l T^j \\ &= \frac{1}{2} \left(\partial_j g_{il} T^l T^j + \partial_i g_{jl} T^l T^j - \partial_j g_{il} T^l T^j\right) \\ &= \frac{1}{2} \partial_i g_{jl} T^l T^j \\ &= \frac{1}{2} \partial_i g_{jk} T^j T^k\end{aligned}$$

where in line 4 we raise the index k, in line 5 we exchange the dummy indices jl in the last term, and in line 7 we relabel l and reorder.

2.9.5 Geodesic Equation

1. List the necessary and sufficient conditions for a curve to be geodesic.
Answer: They are:
• The absolute derivative of the unit tangent vector to the curve vanishes identically. This originates from “the most straight path” definition.
• The first variation of the length of the curve vanishes identically. This originates from “the shortest path” definition.
• The unit tangent vector to the curve is a parallel-transported vector field along the curve.[83] This originates from the “parallel-transporting its tangent” definition.
As seen earlier, all these conditions are equivalent and they lead to the same geodesic equation (i.e. $\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda} = 0$) as a formal characterizing criterion for geodesic curve.
2. What is the difference between the geodesics of special relativity and the geodesics of general relativity? Try to justify your answer technically.
Answer: Since the space of special relativity is flat (i.e. Minkowskian 4D spacetime) the geodesic paths are straight lines. On the other hand, the space of general relativity is curved (i.e. Riemannian 4D spacetime) and hence the geodesic paths are generally twisted curves. This can be seen technically

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[83] Or more generally: the curve has a constant direction with respect to a parallel-transported vector field along the curve.

from the geodesic equation (i.e. $\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0$) because for the flat space of special relativity (assuming Cartesian spatial coordinates) we have $\Gamma^\alpha_{\beta\gamma} = 0$ and hence the geodesic equation becomes $\frac{d^2x^\alpha}{d\lambda^2} = 0$ whose solution is straight line, while for the curved space of general relativity we have $\Gamma^\alpha_{\beta\gamma} \neq 0$ in general and hence the solution of the geodesic equation will not be straight line in general. As indicated earlier, the above assumption of Cartesian spatial coordinates does not affect the generality of the results due to the invariance of geodesic as a geometric attribute whose nature (i.e. being straight) is independent of the employed coordinate system.

3. In the last four sections (i.e. § 2.6, § 2.7, § 2.8 and § 2.9) we investigated the topics of Christoffel symbols, absolute derivative, parallel transport and geodesic path. Try to investigate the relation between these topics and how they are linked together. Also, outline how this leads to the derivation of the geodesic equation.

Answer: The partial derivatives of the basis vectors of a given coordinate system of a given space are vectors that belong to the space and hence they can be expressed as linear combination of the basis vectors of the space. Accordingly, we obtain the following relations: $\partial_j \mathbf{E}_i = \Gamma^k_{ij}\mathbf{E}_k$ and $\partial_j \mathbf{E}^i = -\Gamma^i_{jk}\mathbf{E}^k$ which we consider as the fundamental formal definition of the Christoffel symbols.

Now, let have a coordinate-dependent vector field $\mathbf{A} \equiv A^i\mathbf{E}_i$ in the above space and we want to find out how this vector field varies along a given parameterized curve $C(t)$ where t is a variable parameter that identifies the coordinates of the points of C.[84] To do this we take the derivative of this vector because the derivative (as a rate of change with respect to t) quantifies this variation, that is:

$$\frac{d\mathbf{A}}{dt} \equiv \frac{\delta\mathbf{A}}{\delta t} = \left(\frac{dA^i}{dt} + A^k\Gamma^i_{jk}\frac{dx^j}{dt}\right)\mathbf{E}_i \qquad (12)$$

where the equivalence $\frac{d\mathbf{A}}{dt} \equiv \frac{\delta\mathbf{A}}{\delta t}$ is based on having a general coordinate system whose basis vectors are variable functions of coordinates. As we see, the Christoffel symbols Γ^i_{jk} are needed to find the absolute derivative, and this explains the relation between absolute derivative and Christoffel symbols. So, we established the relation between the Christoffel symbols of a given coordinate system of a given space and the absolute derivative of a vector in that space where this vector is expressed in terms of the basis vectors of the given coordinate system. Now, we need to establish the relation between the absolute derivative and parallel transport. This relation should be obvious because in parallel transport we move a vector along a parameterized curve[85] while keeping it constant in magnitude and direction and hence the detection and quantification of the variation of this vector (through its absolute derivative) is essential for characterizing parallel transport.

Finally, we should establish the relation between parallel transport and geodesic path. As explained earlier, geodesic path is characterized by the property that the tangent vector to the geodesic path remains tangent when it is parallel-transported along the path. This means that the unit tangent vector $\mathbf{T}$ of the geodesic path does not vary (i.e. it remains unit tangent) during the process of parallel transport, i.e. the variation of the unit tangent vector (where this variation is measured by the absolute derivative) vanishes identically along the entire curve. In other words, the absolute derivative of the unit tangent vector is zero identically, that is:

$$\frac{\delta T^i}{\delta t} \equiv \frac{dT^i}{dt} + T^k\Gamma^i_{jk}\frac{dx^j}{dt} = 0$$

Now, the unit tangent vector of a curve is given by $T^i = \frac{dx^i}{d\lambda}$ where $x^i = x^i(\lambda)$ represents the coordinates of the curve and λ (which replaces t now) is a natural parameter for the curve. On substituting this into the last equation we get:

$$\frac{d^2x^i}{d\lambda^2} + \Gamma^i_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} = 0$$

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[84] Vector field is a typical example of tensor field, and hence the obtained results can be trivially generalized to higher rank tensor fields.

[85] We note that "moving a vector along a parameterized curve" is equivalent to observing and detecting the variation of a vector field along the curve.

which is the geodesic equation. So, the geodesic equation is a mathematical statement of the fact that the absolute derivative of the unit tangent vector of a geodesic curve vanishes identically where this "vanishes identically" condition comes from the definition of parallel transport (i.e. moving a vector while keeping it constant in magnitude and direction).

4. Show that in the Minkowski spacetime the geodesics of massive objects are described by the conditions: $\frac{d^2t}{d\tau^2} = 0$ and $\frac{d^2\mathbf{r}}{d\tau^2} = \mathbf{0}$. What this means?
   Answer: The Minkowski spacetime is flat and hence $\Gamma^{\alpha}_{\beta\gamma} = 0$ (assuming Cartesian spatial coordinates). Therefore, the geodesic equation of massive objects becomes $\frac{d^2x^\alpha}{d\tau^2} = 0$ where we use the proper time as affine parameter (see Problems). The equation $\frac{d^2x^\alpha}{d\tau^2} = 0$ means $\frac{d^2t}{d\tau^2} = 0$ for $\alpha = 0$ (since $x^0 = ct$ and c is non-zero constant) and $\frac{d^2\mathbf{r}}{d\tau^2} = \mathbf{0}$ for $\alpha = 1, 2, 3$.
   The meaning of these conditions is that the world lines of free massive objects in the Minkowski spacetime are straight lines in the spacetime (i.e. both in space and in time) which is equivalent to having constant velocity as required by Newton's first law.
5. Why the tensorial nature of the geodesic equation complicates its solution?
   Answer: The tensorial nature means that we are actually required to solve a system of equations rather than a single equation. In the case of the 4D spacetime of general relativity the system consists of 4 equations because the geodesic equation is a vectorial equation since it has a single free index (i.e. α in the form given in the text).
6. Justify the invariance of geodesic paths across various coordinate systems and its independence of the type of representation and parameterization of the space and curve.
   Answer: The invariance of geodesic should be obvious because it is a real geometric property and hence it should not depend on our coordinate systems which have only conventional (rather than real) value. The invariance can also be inferred formally from the tensorial nature of the geodesic equation. This similarly applies to the type of representation and parameterization.
7. What is the relation between the tensorial geodesic equation and the geodesic trajectory of an object (massive or massless) in spacetime?
   Answer: The tensorial geodesic equation is a rank-1 tensor relation (i.e. vector) and hence it represents a system of 4 differential equations that the geodesic trajectory of the object should satisfy. Hence, to find the geodesic trajectory we solve this system of equations simultaneously subject to certain conditions. For any given initial conditions for the object in the spacetime the solution is unique, i.e. the trajectory is completely determined. The required initial conditions are the 4-position and the 4-velocity of the object. The existence and uniqueness of the solution are guaranteed by the theorems of differential geometry. Some of these theorems are given in B2 for 2D spaces, e.g. "for any specific point P on a surface S of class C^3, and for any tangent vector $\mathbf{v}$ in the tangent space of S at P, there exists a geodesic curve on the surface in the direction of $\mathbf{v}$ that passes through P".
   Note: regarding the required initial conditions, the 4-position condition arises from the necessity of identifying the location of the object in the spacetime while the 4-velocity condition arises from the necessity of identifying the tangential direction of the object in the spacetime. The latter should be obvious in the case of massive objects where the 4-velocity is given by $dx^\mu/d\tau$ (with the proper time τ being used as an affine parameter in the tangent equation $dx^\mu/d\lambda$). From a formal perspective, the requirement of these two conditions (i.e. position and velocity) arises from the fact that the geodesic equation is a second order differential equation and hence two initial conditions are required to have a unique solution (noting that these conditions are 4D vectorial relations like the geodesic equation itself).
8. The motion of free objects (i.e. free of forces other than gravity) is subject to the geodesic equation. How will this change when the objects are not free (i.e. they are subject to non-gravitational forces)?
   Answer: According to general relativity, the motion of non-free objects is also subject to the geodesic equation (i.e. $\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^{\alpha}_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0$) but with the addition of a force source term to the right hand side to account for the effect of non-gravitational forces.[86] In brief, the geodesic equation applies to

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[86] However, in this case the equation should not be labeled as "geodesic equation" since the motion described by the modified

both free and non-free objects but with vanishing source term for free objects and with non-vanishing source term for non-free objects. Accordingly, the geodesic equation (without and with force term) may be seen as the general relativistic equivalent of the classical Newton's laws of motion (first and second respectively). Also see exercise 8 of § 2.9 and exercise 2 of § 7.9.

2.10 Riemann-Christoffel Curvature Tensor

1. What are the physical dimensions of the Riemann-Christoffel curvature tensor?
Answer: The Riemann-Christoffel curvature tensor is given by:

$$R_{ijkl} = \partial_k \left[jl, i\right] - \partial_l \left[jk, i\right] + \left[il, r\right] \Gamma^r_{jk} - \left[ik, r\right] \Gamma^r_{jl}$$

Now, if we note that the dimensions of both the differential operator (e.g. $\partial_k \equiv \frac{\partial}{\partial x^k}$) and the Christoffel symbols are reciprocal length (see § 2.6)[87] then we can conclude that the physical dimensions of the Riemann-Christoffel curvature tensor are reciprocal area (e.g. m^{-2}).

2. Make a list of all the main properties of the Riemann-Christoffel curvature tensor (i.e. rank, type, symmetry, etc.).
Answer: Some of these properties are:
• It is absolute rank-4 tensor.
• It is commonly used as covariant of type $(0, 4)$ or as mixed of type $(1, 3)$.
• The covariant type is anti-symmetric in its first two indices and in its last two indices and block symmetric in the first and second pairs of indices, while the mixed type is anti-symmetric in its last two indices.
• It depends only on the metric tensor and hence it is intrinsic to the space.
• It characterizes the space and hence it is used for example as a test for the intrinsic curvature of the space since it vanishes identically *iff* the space is flat (and does not vanish identically *iff* the space is curved).
• When it vanishes the covariant differential operators become commutative.
• It has the physical dimensions of reciprocal area.
• In nD space it has n^4 components and $\frac{n^2(n^2-1)}{12}$ independent non-vanishing components.

3. When the Riemann-Christoffel curvature tensor vanishes the covariant differential operators become commutative. Justify.
Answer: This can be easily justified by the fact that when it vanishes identically then the space is flat and hence it can be coordinated by a Cartesian system and therefore the covariant differential operators become ordinary partial differential operators which are commutative. Now, since the commutator of the differential operators is a tensor (noting that the ordinary derivative of a tensor in a Cartesian system is a tensor) then this should apply in any coordinate system.
More formally, the relation $A_{j;kl} - A_{j;lk} = R^i_{\ jkl} A_i$ (which we established in B3) clearly shows that if $R^i_{\ jkl}$ vanishes then the commutator $A_{j;kl} - A_{j;lk}$ must also vanish, i.e. the covariant differential operators become commutative.

4. Show that $R_{ijkl} = 0$ if the Christoffel symbols vanish identically. What is the implication of this?
Answer: The Riemann-Christoffel curvature tensor is given by:

$$R_{ijkl} = \partial_k \left[jl, i\right] - \partial_l \left[jk, i\right] + \left[il, r\right] \Gamma^r_{jk} - \left[ik, r\right] \Gamma^r_{jl}$$

and hence if the Christoffel symbols vanish identically then $R_{ijkl} = 0$.
The obvious implication is that the Riemann-Christoffel curvature tensor vanishes identically in flat space. This is because the vanishing of the Christoffel symbols is a sufficient (but not necessary) condition for space flatness (see exercise 8 of § 2.6).

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equation will not follow a geodesic trajectory.

[87] We should remind the reader of the non-generality of this although the result still holds assuming a suitable type of coordinate system.

5. Give a proof for the second Bianchi identity.
Answer: The simplest proof is based on the use of geodesic coordinates (see § 2.1) with the covariant differentiation of the Riemann-Christoffel curvature tensor at the pole. On applying cyclic interchange of the last three indices of the covariant derivative and adding the resulting tensors the identity can be obtained. The generality of the result over the entire space will then be obtained by the premise that the choice of the pole is arbitrary (i.e. any point in the space can be chosen as a pole of the system) and hence the result should be valid at any point in the space. The generality of this general result in any coordinate system should then be obtained from its tensorial nature and the value-invariance (or constancy) of the zero tensor across all coordinate systems.
In the following, we cast the above argument in a more technical form. On covariant-differentiating Eq. 77 (in the text) we get:

$$R^i_{\ jkl;m} = \left(\partial_k \Gamma^i_{jl}\right)_{;m} - \left(\partial_l \Gamma^i_{jk}\right)_{;m} + \left(\Gamma^r_{jl}\right)_{;m} \Gamma^i_{rk} + \Gamma^r_{jl} \left(\Gamma^i_{rk}\right)_{;m} - \left(\Gamma^r_{jk}\right)_{;m} \Gamma^i_{rl} - \Gamma^r_{jk} \left(\Gamma^i_{rl}\right)_{;m} \qquad (13)$$

Now, at the pole of geodesic coordinate system the Christoffel symbols vanish (although their derivatives do not vanish in general) and the covariant derivative becomes partial derivative. Therefore, the last equation becomes:

$$R^i_{\ jkl,m} = \partial_m \partial_k \Gamma^i_{jl} - \partial_m \partial_l \Gamma^i_{jk}$$

On cycling the klm indices we get:

$$\begin{aligned} R^i_{\ jmk,l} &= \partial_l \partial_m \Gamma^i_{jk} - \partial_l \partial_k \Gamma^i_{jm} \\ R^i_{\ jlm,k} &= \partial_k \partial_l \Gamma^i_{jm} - \partial_k \partial_m \Gamma^i_{jl} \end{aligned}$$

On adding the last three equations (noting that the partial derivative operators are commutative) we get:

$$R^i_{\ jkl,m} + R^i_{\ jmk,l} + R^i_{\ jlm,k} = 0$$

Because the arbitrarily chosen pole can represent any point in the space the last equation is valid over the entire space. Moreover, the last equation is a tensorial relation (since it is obtained from Eq. 13 which is tensorial) and hence it must be valid in any coordinate system where the general validity in general coordinate systems requires the partial derivative to become covariant derivative, that is:

$$R^i_{\ jkl;m} + R^i_{\ jmk;l} + R^i_{\ jlm;k} = 0$$

which is the second Bianchi identity of the second kind. The second Bianchi identity of the first kind is obtained by lowering the index i, that is:

$$R_{ijkl;m} + R_{ijmk;l} + R_{ijlm;k} = 0$$

6. The first Bianchi identity $R_{ijkl} + R_{iljk} + R_{iklj} = 0$ may be expressed as $R_{i[jkl]} = 0$ where the square brackets [ ] mean anti-symmetrization in the enclosed indices. Justify.
Answer: We have:

$$\begin{aligned} R_{i[jkl]} &= 0 \\ \frac{1}{3!}\left(R_{ijkl} + R_{iljk} + R_{iklj} - R_{ijlk} - R_{ikjl} - R_{ilkj}\right) &= 0 \\ R_{ijkl} + R_{iljk} + R_{iklj} - R_{ijlk} - R_{ikjl} - R_{ilkj} &= 0 \\ R_{ijkl} + R_{iljk} + R_{iklj} + R_{ijkl} + R_{iklj} + R_{iljk} &= 0 \\ 2R_{ijkl} + 2R_{iljk} + 2R_{iklj} &= 0 \\ R_{ijkl} + R_{iljk} + R_{iklj} &= 0 \end{aligned}$$

where in line 2 we expand the anti-symmetric part in the three enclosed indices (see § 3.1.5 of B3), while in line 4 we use the anti-symmetry property of the Riemann-Christoffel curvature tensor in its last two indices.

2.11 Ricci Curvature Tensor

1. What are the physical dimensions of the Ricci curvature tensor?
   Answer: The Ricci curvature tensor is a contraction of the Riemann-Christoffel curvature tensor and hence it also has the dimensions of reciprocal area (e.g. m^{-2}; see § 2.10). We note that the contraction of a tensor is obtained by summing components of the original tensor after unifying the contracted indices and therefore the physical dimensions of the contracted tensor are the same as the physical dimensions of the original tensor. More technically, the contraction is conducted by inner producting the metric tensor (or the Kronecker delta which is also a metric) with the contracted tensor (or mother tensor) and hence the resulting tensor (or daughter tensor) should have the same physical dimensionality as the mother tensor because the metric tensor is dimensionless (see § 2.5).
2. What are the number of components and the number of independent components of the Ricci curvature tensor in the spacetime of general relativity?
   Answer: The Ricci curvature tensor is a rank-2 symmetric tensor in 4D spacetime and hence it has $4^2 = 16$ components and $\frac{4(4+1)}{2} = 10$ independent components.
3. Make a list of the main properties of the Ricci curvature tensor (rank, type, symmetry, etc.).
   Answer: The main properties of the Ricci curvature tensor are:
   - It is derived from the Riemann-Christoffel curvature tensor by contracting two indices.
   - It is absolute rank-2 tensor.
   - It depends only on the metric tensor and hence it is intrinsic to the space.
   - It characterizes the geometry of space and expresses its curvature (partly).
   - It is commonly used as covariant (first kind) of type $(0, 2)$ or mixed (second kind) of type $(1, 1)$.
   - The covariant type is symmetric.
   - It has the physical dimensions of reciprocal area.
   - It vanishes identically in flat space.
   - In nD space it has n^2 components and $\frac{n(n+1)}{2}$ independent components.
4. The first kind of the Ricci curvature tensor may also be obtained by contracting the contravariant index of the Riemann-Christoffel curvature tensor of the second kind with its covariant index before the last. Comment on this.
   Answer: Yes, but in this case the sign will be reversed because $R^{a}{}_{iaj} = -R^{a}{}_{ija}$ due to the anti-symmetry in the last two indices.
5. Show that $R_{ij} = 0$ identically is a necessary but not sufficient condition for a space to be flat.
   Answer: We have $R_{ij} = R^{b}{}_{ijb} = g^{ab}R_{aijb}$. So, if $R_{aijb} = 0$ identically (which is a necessary and sufficient condition for a space to be flat) then $R_{ij} = 0$. Hence, $R_{ij} = 0$ is a necessary condition for space flatness. On the other hand, $R_{ij} = 0$ identically does not imply $R_{aijb} = 0$ identically. Hence, $R_{ij} = 0$ is not a sufficient condition for space flatness. The justification of all this is that R_{ij} is a contraction of R_{aijb} (i.e. the mixed form of R_{aijb}) and hence the components of R_{ij} are obtained by summing some components of R_{aijb}. So, if $R_{aijb} = 0$ (i.e. all the components of R_{aijb} vanish) then $R_{ij} = 0$ but if $R_{ij} = 0$ it is not necessary that $R_{aijb} = 0$ because the components of R_{aijb} may not be zero but the sum(s) of some of these components (which form the contraction) could be zero. For example, if A_i^j is a rank-2 tensor represented by the matrix $\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ then $A_i^j \neq 0$ but $A_i^i = 1 - 1 = 0$.[88]
   So in brief, from a purely geometric perspective (i.e. regardless of the physical perspective as represented by the Field Equation of general relativity) $R_{aijb} = 0$ identically is a necessary and sufficient condition for space flatness, while $R_{ij} = 0$ identically is a necessary but not sufficient condition for space flatness.

2.12 Ricci Curvature Scalar

1. What are the physical dimensions of the Ricci curvature scalar?

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[88] A real physical example is the energy-momentum tensor in spacetime occupied by electromagnetic field only where the tensor is not zero but its trace is zero (see exercises 5 and 6 of § 2.14).

Answer: The Ricci curvature scalar is a contraction of the Ricci curvature tensor and hence it also has the dimensions of reciprocal area (e.g. m^{-2}; see exercise 1 of § 2.11).

2. Make a list of the main properties of the Ricci curvature scalar.
Answer: The main properties of the Ricci curvature scalar are:
- It is derived from the Ricci curvature tensor (and ultimately from the Riemann-Christoffel curvature tensor) by contracting its two indices.
- It is absolute rank-0 tensor (i.e. scalar). Hence, it has a single component in any space (of any dimension).
- It depends only on the metric tensor and hence it is intrinsic to the space.
- It characterizes the geometry of space and expresses its curvature (partly).
- It has the physical dimensions of reciprocal area.
- It vanishes identically in flat space.

3. Show that $R = 0$ identically is a necessary but not sufficient condition for a space to be flat.
Answer: We have $R = R^{j}_{j} = g^{ij}R_{ij} = g^{ij}R^{b}{}_{ijb} = g^{ij}g^{ab}R_{aijb}$. So, if $R_{aijb} = 0$ identically (which is a necessary and sufficient condition for a space to be flat) then $R = 0$ identically. The rest of the argument is similar to the argument in exercise 5 of § 2.11 about the provision $R_{ij} = 0$ as being a necessary but not sufficient condition for space flatness.

4. Compare the curvature information contained in the Riemann-Christoffel curvature tensor, the Ricci curvature tensor and the Ricci curvature scalar.
Answer: Because the Ricci curvature tensor is a contraction of the Riemann-Christoffel curvature tensor then it contains less curvature information than the Riemann-Christoffel curvature tensor. Similarly, because the Ricci curvature scalar is a contraction of the Ricci curvature tensor then it contains less curvature information than the Ricci curvature tensor and the Riemann-Christoffel curvature tensor. In fact, this answer should justify the fact that the vanishing of the Riemann-Christoffel curvature tensor is a necessary and sufficient condition for space flatness, while the vanishing of the Ricci curvature tensor and the vanishing of the Ricci curvature scalar are necessary but not sufficient conditions for space flatness.[89]

2.13 Einstein Tensor

1. Show that the physical dimensions of the Einstein tensor are reciprocal area.
Answer: The Einstein tensor is given by $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$. Now, since $R_{\mu\nu}$ and R have the dimensions of reciprocal area (see § 2.11 and 2.12) while $g_{\mu\nu}$ is dimensionless (see § 2.5) then the physical dimensions of the Einstein tensor are reciprocal area (e.g. m^{-2}).

2. In flat space the Einstein tensor is zero. Explain why.
Answer: In flat space the Riemann-Christoffel curvature tensor is zero and hence the Ricci curvature tensor and the Ricci curvature scalar are zero (see exercise 5 of § 2.11 and exercise 3 of § 2.12). Therefore, the Einstein tensor must also be zero according to the equation $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$.

3. What the identity $G^{\mu\nu}_{\ \ ;\nu} = 0$ means?
Answer: It means that the divergence of the Einstein tensor is identically zero throughout the space and for any metric.

4. Prove that the divergence of Einstein tensor is identically zero (i.e. $G^{\mu\nu}_{\ \ ;\nu} = 0$) using the second Bianchi identity.
Answer: We have:

$$\begin{aligned}
R_{ijkl;m} + R_{ijmk;l} + R_{ijlm;k} &= 0 \\
-R_{ijlk;m} + R_{ijmk;l} + R_{jiml;k} &= 0 \\
g^{ik}\left(-R_{ijlk;m} + R_{ijmk;l} + R_{jiml;k}\right) &= 0 \\
-R^{k}_{jlk;m} + R^{k}_{jmk;l} + R_{j\bullet ml;k}^{\ k} &= 0
\end{aligned}$$

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[89] The reader, however, should notice that the discussion here is from a purely differential geometric perspective and not from a general relativistic perspective (i.e. assuming the validity of the Field Equation). Further investigations of this issue (and related issues) will follow.

$$
\begin{aligned}
g^{jl}\left(-R^{k}_{jlk;m}+R^{k}_{jmk;l}+R_{j\bullet ml;k}^{\;k}\right) &= 0\\
-R^{kl}{}_{lk;m}+R^{kl}{}_{mk;l}+R^{lk}{}_{ml;k} &= 0\\
-R^{l}{}_{l;m}+R^{l}{}_{m;l}+R^{k}{}_{m;k} &= 0\\
-R_{;m}+R^{l}{}_{m;l}+R^{k}{}_{m;k} &= 0\\
-R_{;m}+R^{k}{}_{m;k}+R^{k}{}_{m;k} &= 0\\
-R_{;m}+2R^{k}{}_{m;k} &= 0\\
R^{k}{}_{m;k}-\frac{1}{2}R_{;m} &= 0\\
R^{k}{}_{m;k}-\frac{1}{2}\delta^{k}{}_{m}R_{;k} &= 0\\
R^{k}{}_{m;k}-\frac{1}{2}g^{k}{}_{m}R_{;k} &= 0\\
R^{k}{}_{m;k}-\left(\frac{1}{2}g^{k}{}_{m}R\right)_{;k} &= 0\\
\left(R^{k}{}_{m}-\frac{1}{2}g^{k}{}_{m}R\right)_{;k} &= 0\\
\left(g^{lm}R^{k}{}_{m}-\frac{1}{2}g^{lm}g^{k}{}_{m}R\right)_{;k} &= 0\\
\left(R^{kl}-\frac{1}{2}g^{kl}R\right)_{;k} &= 0\\
G^{kl}_{\;;k} &= 0\\
G^{lk}_{\;;k} &= 0\\
G^{\mu\nu}_{\;;\nu} &= 0
\end{aligned}
$$

where line 1 is the second Bianchi identity, line 2 is the anti-symmetry property of the Riemann-Christoffel curvature tensor in its first two indices and in its last two indices, in lines 3 and 4 we raise the index i, in lines 5 and 6 we raise the index j, in line 7 we carry out contraction between the first and fourth indices in all terms, in line 8 we carry out contraction between the first and second indices in the first term using the definition of the Ricci curvature scalar, in line 9 we relabel the dummy index in the second term, in line 10 we add the last two terms, in line 11 we divide by 2, in line 12 we use the Kronecker delta as an index replacement operator, in line 13 we use the fact that $g^k_m \equiv \delta^k_m$, in line 14 we use the fact that the metric tensor is like a constant with respect to covariant differentiation (see § 2.7), in line 15 we use the fact that the sum rule of differentiation applies to covariant differentiation like ordinary differentiation (see § 2.7), in line 16 we multiply with g^{lm} using the fact that the metric tensor is like a constant with respect to covariant differentiation, in line 17 we raise the index m, in line 18 we use the definition of Einstein tensor, in line 19 we use the symmetry of Einstein tensor, and in line 20 we change the Latin indices (which we used for ease of typesetting) to Greek indices.

We note that the divergence of the mixed type Einstein tensor is also zero (i.e. $G^{\nu}_{\mu;\nu}=0$) as can be seen from line 15.

5. Show that $g^{\nu\alpha}G_{\alpha\mu;\nu}=0$.
   Answer: From the previous exercise we have $G^{\nu}_{\mu;\nu}=0$. Moreover, $G^{\nu}_{\mu;\nu}=g^{\nu\alpha}G_{\alpha\mu;\nu}$. Hence, $g^{\nu\alpha}G_{\alpha\mu;\nu}=0$.
6. Show that $G=-R$ where G is the trace of the Einstein tensor and R is the Ricci curvature scalar.
   Answer: We have:

$$
\begin{aligned}
G_{\mu\nu} &= R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\\
g^{\mu\nu}G_{\mu\nu} &= g^{\mu\nu}R_{\mu\nu}-\frac{1}{2}g^{\mu\nu}g_{\mu\nu}R
\end{aligned}
$$

$$\begin{aligned} G^{\nu}_{\ \nu} &= R^{\nu}_{\ \nu} - \frac{1}{2}\delta^{\nu}_{\ \nu}R \\ G &= R - \frac{1}{2}\delta^{\nu}_{\ \nu}R \\ G &= R - \frac{1}{2}4R \\ G &= -R \end{aligned}$$

where in line 1 we use the definition of Einstein tensor, in line 2 we multiply the equation with $g^{\mu\nu}$, in line 3 we contract the indices and use $g^{\nu}_{\ \nu} \equiv \delta^{\nu}_{\ \nu}$, in line 4 we use the definition of G and R as traces of $G^{\mu}_{\ \nu}$ and $R^{\mu}_{\ \nu}$, and in line 5 we use $\delta^{\nu}_{\ \nu} = \delta^{0}_{\ 0} + \delta^{1}_{\ 1} + \delta^{2}_{\ 2} + \delta^{3}_{\ 3} = 4$.
Note: this may also be shown in a rather short way as:

$$G \equiv G^{\nu}_{\ \nu} = R^{\nu}_{\ \nu} - \frac{1}{2}\delta^{\nu}_{\ \nu}R = R - 2R = -R$$

where step 1 is the definition of G as trace of $G^{\mu}_{\ \nu}$, step 2 is the definition of Einstein tensor, and step 3 is the definition of R as trace of $R^{\mu}_{\ \nu}$ plus the relation $\delta^{\nu}_{\ \nu} = 4$.

7. Show that $G_{\mu\nu} = 0$ *iff* $R_{\mu\nu} = 0$.
Answer: We have two parts:
• If $R_{\mu\nu} = 0$ then $G_{\mu\nu} = 0$: this is obvious because if $R_{\mu\nu} = 0$ then $R = R^{\nu}_{\ \nu} = g^{\mu\nu}R_{\mu\nu} = 0$ and hence $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 - 0 = 0$.
• If $G_{\mu\nu} = 0$ then $R_{\mu\nu} = 0$: this is because in this case we have:

$$\begin{aligned} G_{\mu\nu} &= 0 \qquad (14) \\ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= 0 \\ g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}g^{\mu\nu}g_{\mu\nu}R &= 0 \\ R - \frac{1}{2}\delta^{\nu}_{\ \nu}R &= 0 \\ R\left(1 - \frac{1}{2}4\right) &= 0 \\ R &= 0 \qquad (15) \end{aligned}$$

where in line 2 we use the definition of Einstein tensor, in line 3 we multiply by $g^{\mu\nu}$, in line 4 we contract the indices using the definition of R as trace of $R^{\nu}_{\ \mu}$ and $g^{\mu\nu}g_{\mu\nu} = g^{\nu}_{\ \nu} \equiv \delta^{\nu}_{\ \nu}$, and in line 5 we use $\delta^{\nu}_{\ \nu} = \delta^{0}_{\ 0} + \delta^{1}_{\ 1} + \delta^{2}_{\ 2} + \delta^{3}_{\ 3} = 4$. Accordingly:

$$\begin{aligned} G_{\mu\nu} &= 0 \\ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= 0 \\ R_{\mu\nu} &= 0 \end{aligned}$$

where in line 2 we use the definition of Einstein tensor, and in line 3 we use $R = 0$ which we obtained earlier (see Eq. 15).
Note: this may also be shown in a rather short way as follows:
If $R_{\mu\nu} = 0$ then $R = 0$ and hence $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 - 0 = 0$.
If $G_{\mu\nu} = 0$ then $G = 0$ and hence according to exercise 6 $R = 0$ and therefore $R_{\mu\nu} = 0$ (according to the definition of Einstein tensor).

8. Show the following: (**a**) $G = 0$ *iff* $R = 0$ and (**b**) if $G_{\mu\nu} = 0$ then $R = 0$.
Answer:
(**a**) This can be concluded from $G = -R$ which we obtained in exercise 6.
(**b**) This is shown in the previous exercise (Eqs. 14 and 15).

9. Compare the curvature information contained in the Riemann-Christoffel curvature tensor and the Einstein tensor. What is the implication of this?
Answer: Because the Einstein tensor is synthesized from the Ricci curvature tensor and scalar (as well as the metric tensor) then it contains less curvature information than the Riemann-Christoffel curvature tensor. The implication is that since the Einstein tensor represents the geometric component of the Field Equation (see § 3.2) then the gravity of general relativity is correlated to a certain part of the curvature of spacetime rather than the entire curvature of spacetime.
Note: the inclusion of the metric tensor in the definition of the Einstein tensor does not recover the missing part of the curvature information that is contained in the Riemann-Christoffel curvature tensor because although the metric tensor represents the geometry of the spacetime it does not represent the curvature directly. The fact that the Einstein tensor does not include full curvature information of the spacetime may be supported by the following facts:
• The full curvature information are contained in the rank-4 Riemann-Christoffel curvature tensor which has 20 independent components and this cannot be contained in the rank-2 Einstein tensor which has only 10 independent components.
• As we will see later, for the vacuum problems[90] we have $G_{\mu\nu} = 0$ (and equivalently $R_{\mu\nu} = 0$ as shown in exercise 7) although the spacetime represented by this tensor is generally curved which means that the curvature of spacetime (in the vacuum region) is actually represented by the Riemann-Christoffel curvature tensor (since flatness implies $R_{\mu\nu\rho\omega} = 0$ and curvedness implies $R_{\mu\nu\rho\omega} \neq 0$) but is not represented by the Einstein tensor (since $G_{\mu\nu} = 0$ does not imply flatness because this condition can belong to a curved region as well as a flat region). In brief, the condition $G_{\mu\nu} = 0$ (and equivalently $R_{\mu\nu} = 0$) on its own is not informative about the flatness/curvedness of the vacuum region because $G_{\mu\nu}$ does not contain full curvature information, but the condition $R_{\mu\nu\rho\omega} = 0$ on its own is fully informative about the flatness/curvedness of the vacuum region because $R_{\mu\nu\rho\omega} = 0$ contains full curvature information. We should note that although the vacuum equation $G_{\mu\nu} = 0$ usually leads to a solution (which may be interpreted as containing all the required curvature information) the generality of this is not obvious. In fact, the representation of all the missing curvature information in the condition $G_{\mu\nu} = 0$ (and hence in the solution of this equation) is not obvious or guaranteed. Moreover, even if the existence of a solution that completely represents all the curvature information is guaranteed theoretically, from a practical perspective obtaining this solution is not guaranteed due to the ambiguity of the vacuum equation (see § 10.1.16). Also refer to the questions (Problems and Exercises) of § 3.2.

2.14 Energy-Momentum Tensor

1. Why the energy-momentum tensor has 16 components with only 10 of these being independent?
Answer: The energy-momentum tensor is a rank-2 symmetric tensor in 4D spacetime. Since a rank-2 symmetric tensor in an nD space has n^2 components and $\left(n^2 + n\right)/2$ independent components (see § 2.2), the given numbers (i.e. 16 and 10) are justified.
2. What are the physical dimensions of the energy-momentum tensor? Provide detailed explanation.
Answer: We defined this tensor generically as the flux of 4-momentum where flux means "per area per time". Now, if we multiply and divide the physical dimensions by length (which does not affect the overall physical dimensions of the tensor) then the physical dimensions of the tensor become "momentum times speed per volume". Since "momentum times speed" is energy, the physical dimensions of the tensor can be expressed as energy per volume (e.g. $\mathrm{J/m^3} = \mathrm{kg\,m^{-1}\,s^{-2}}$) as stated in the text.
3. Make a list of the main properties of the energy-momentum tensor.
Answer: The main properties are:
• It describes the distribution and flow of energy and momentum in the 4D spacetime of general relativity[91] and hence it represents the physical part in the Field Equation.

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[90] "Vacuum problem" means gravitational problem related to a region of spacetime where there is no matter or energy.

[91] The restriction to general relativity is because this is our subject of investigation; otherwise the energy-momentum tensor is more general.

- It is rank-2 symmetric tensor and hence in the 4D spacetime it has 16 components 10 of which are independent.
- It has the physical dimensions of energy density (e.g. J/m^3).
- Its divergence is zero.
- When it vanishes throughout the spacetime, the spacetime of general relativity reduces to the spacetime of special relativity, i.e. it becomes Minkowskian flat space.[92]
- Its components are as follows: T^{00} represents energy density, $T^{0i} = T^{i0}$ represents momentum flow, T^{ii} represents normal stress and T^{ij} $(i \neq j)$ represents shear stress (where $i, j = 1, 2, 3$).

4. Show that the divergence of the energy-momentum tensor is zero (i.e. $T^{\mu\nu}_{\ \ ;\nu} = 0$) assuming that the Field Equation is given as a postulate.
Answer: If we postulate the Field Equation then we have (see § 3.2):

$$\begin{aligned}\kappa T^{\mu\nu} &= G^{\mu\nu} \\ \left(\kappa T^{\mu\nu}\right)_{;\nu} &= G^{\mu\nu}_{\ \ ;\nu} \\ \left(\kappa T^{\mu\nu}\right)_{;\nu} &= 0 \\ \kappa T^{\mu\nu}_{\ \ ;\nu} &= 0 \\ T^{\mu\nu}_{\ \ ;\nu} &= 0\end{aligned}$$

where line 1 is the Field Equation, in line 2 we take the divergence of the two sides, in line 3 we use the fact that the divergence of the Einstein tensor is zero (see § 2.13), in line 4 we use the fact that κ is a constant, and in line 5 we use the fact that $\kappa \neq 0$.

5. What is the energy-momentum tensor of electromagnetic field? What is the significance of this?
Answer: In a region of spacetime occupied by electromagnetic field only (i.e. void of matter) the energy-momentum tensor $T^{\mu\nu}$ is given by:

$$T^{\mu\nu} = A\left(S^{\mu}_{\ \alpha}S^{\nu\alpha} - \frac{1}{4}g^{\mu\nu}S_{\alpha\beta}S^{\alpha\beta}\right)$$

where the indexed S is the electromagnetic field strength tensor (see B4), $g^{\mu\nu}$ is the metric tensor, and A is a constant.
The significance is that according to general relativity even electromagnetic energy can be a source of gravity like matter (see for example § 4.3).

6. Show that the trace of the energy-momentum tensor of electromagnetic field is zero.
Answer: We have:

$$\begin{aligned}T^{\mu\nu} &= A\left(S^{\mu}_{\ \alpha}S^{\nu\alpha} - \frac{1}{4}g^{\mu\nu}S_{\alpha\beta}S^{\alpha\beta}\right) \\ g_{\mu\nu}T^{\mu\nu} &= A\left(g_{\mu\nu}S^{\mu}_{\ \alpha}S^{\nu\alpha} - \frac{1}{4}g_{\mu\nu}g^{\mu\nu}S_{\alpha\beta}S^{\alpha\beta}\right) \\ T^{\nu}_{\nu} &= A\left(S_{\nu\alpha}S^{\nu\alpha} - \frac{1}{4}\delta^{\nu}_{\nu}S_{\alpha\beta}S^{\alpha\beta}\right) \\ T &= A\left(S_{\alpha\beta}S^{\alpha\beta} - S_{\alpha\beta}S^{\alpha\beta}\right) \\ T &= 0\end{aligned}$$

where in line 4 we relabel the dummy indices in the first term and use $\delta^{\nu}_{\nu} = \delta^0_0 + \delta^1_1 + \delta^2_2 + \delta^3_3 = 4$.

7. What is the state of the Ricci curvature tensor $R_{\mu\nu}$ and the Ricci curvature scalar R in the vacuum problems[93] of general relativity? What is the state of the curvature of spacetime in these problems?

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[92] We are ignoring dark energy or including it in the energy-momentum tensor (see § 3.3). We should also note that the above is a requirement of general relativity (as formalized in the Field Equation) and not a geometric requirement.

[93] "Vacuum problem" means gravitational problem related to a region of spacetime where there is no matter or energy. Accordingly, vacuum solutions are solutions of vacuum problems.

Comment on the result.
Answer: In the vacuum problems of general relativity the Ricci curvature tensor and the Ricci curvature scalar should vanish, i.e. $R_{\mu\nu} = 0$ and $R = 0$. This is because in the vacuum problems $T_{\mu\nu} = 0$ and hence according to the Filed Equation $G_{\mu\nu} = 0$ which leads to $R_{\mu\nu} = 0$ and $R = 0$ (according to exercises 7 and 8 of § 2.13).
Vacuum problem in general relativity does not mean that the vacuum region of the spacetime is flat. What it means is that the problem (and hence the solution) corresponds to a region of spacetime where there is no matter or energy (i.e. $T_{\mu\nu} = 0$) and hence $R_{\mu\nu} = 0$ and $R = 0$.
Comment: in the vacuum problems of general relativity we have $R_{\mu\nu} = 0$ and $R = 0$ in the vacuum region but the curvature of spacetime in the vacuum region does not vanish, i.e. $R_{\mu\nu\sigma\omega} \neq 0$. The curvature of the vacuum region should then be attributed to the existence of matter or/and energy somewhere else in the spacetime.
Note: if we consider special relativity as a special case of vacuum problems of general relativity then in this case the relationships $R_{\mu\nu} = 0$ and $R = 0$ apply globally and this implies (according to the Field Equation) $T_{\mu\nu} = 0$ globally which according to the fundamental premise of general relativity (i.e. the curvature of spacetime is caused by the existence of matter in the spacetime) means flat spacetime although from a pure differential geometric perspective the conditions $R_{\mu\nu} = 0$ and $R = 0$ globally do not imply $R_{\mu\nu\sigma\omega} = 0$ globally. This should elucidate our assertion that gravitation in general relativity is related to part of the geometric curvature of the spacetime and not to the entire geometric curvature of the spacetime (see § 10.1.15 and § 10.1.16). So in principle, a general-relativistically flat spacetime (i.e. when $R_{\mu\nu} = T_{\mu\nu} = 0$ globally) may be differential-geometrically curved (since $R_{\mu\nu\sigma\omega}$ may not be zero globally).[94]

8. Justify the premise that the vanishing divergence of the energy-momentum tensor (i.e. $T^{\mu\nu}_{;\nu} = 0$) is a requirement (or an implication) for the conservation of the momentum 4-vector.
Answer: In the text we defined the energy-momentum tensor generically as the tensor whose $T^{\mu\nu}$ component represents the flux of the μ^{th} component of the momentum 4-vector across a surface perpendicular to the ν^{th} coordinate. Now, let assume that we are in the flat spacetime of special relativity (with the Minkowski metric).[95] So, if the μ^{th} component of the momentum 4-vector across a surface perpendicular to the ν^{th} coordinate is conserved then what crosses the surface on one side should emerge on the other side unchanged, and this means that the partial derivative of $T^{\mu\nu}$ with respect to the ν^{th} coordinate is zero, i.e. $T^{\mu\nu}_{,\nu} = 0$. So, if we take this argument to a curved spacetime of general relativity (where partial derivative becomes covariant derivative) then we should have $T^{\mu\nu}_{;\nu} = 0$ (by just replacing the comma with semicolon) which is the required result, i.e. $T^{\mu\nu}_{;\nu} = 0$ is a requirement (or an implication) for the conservation of the momentum 4-vector.
Note 1: the momentum 4-vector represents energy (divided by c) in its temporal component and momentum in its spatial components and hence its conservation means the conservation of energy and momentum. However, the reader should be aware that the conservation of energy and momentum in the spacetime of general relativity holds locally but not globally (see for example § 7.9).
Note 2: in the literature of general relativity there are details about the significance of the vanishing divergence of the energy-momentum tensor with regard to the conservation of energy and momentum where it is claimed that in the curved spacetime of general relativity conservation is not implied by the vanishing of divergence due to the presence of gravity whose energy is not included in the energy-momentum tensor.[96] The reader is referred to the literature for details (most of which are based on personal views and arbitrary claims which can be questioned and challenged, like most aspects

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[94] Could this be one reason for the need of dark energy in general relativity and its applications?

[95] The existence of matter and energy (as implied by the momentum 4-vector) does not contradict the flat nature of the spacetime of special relativity as long as the matter and energy are negligible as a source of gravity. In fact, matter and energy always exist in the spacetime of special relativity (as this is evident from the frequent use of quantities like mass, energy and momentum in the physics of special relativity) but not as a source of gravity. Moreover, the flat spacetime of special relativity does exist *locally* even in the presence of matter and energy as a source of gravity where the spacetime is globally curved and hence it belongs to general relativity.

[96] The equation $T^{\mu\nu}_{;\nu} = 0$ may be required to include even the gravitational energy-momentum contribution to be fully representative of the conservation principle. The technical details of this issue are beyond the scope of this book.

of general relativity, and hence they are not worthwhile to pursue). However, this could be another sign of the mess that is created in general relativity by its unusual paradigms such as the paradigm of curved spacetime as a demonstration (or representation or imitation) of gravity instead of treating gravity as an ordinary physical phenomenon (like any other physical phenomenon) that takes place in flat spacetime (or even in curved spacetime where the curvature is not caused by gravity but it is an inherent property of spacetime and hence it has an equal impact on all physical phenomena whether gravitational or not).

Note 3: expressions like "across a surface" are more general with regard to the state of the μ^{th} component of the 4-momentum as being perpendicular or parallel to the surface (noting that the surface can be tangential to the component of the 4-momentum). So, this should not affect our argument that "the partial derivative of $T^{\mu\nu}$ with respect to the ν^{th} coordinate is zero, i.e. $T^{\mu\nu}_{\ \ ,\nu} = 0$" because the essence of differentiation is to express and quantify the variation in the energy-momentum as a function of the variation of the coordinate regardless of the state of the μ^{th} component as being perpendicular or parallel to the surface. Anyway, the entire argument is supposed to provide rough qualitative reasoning rather than rigorous quantitative formulation and within this limit the argument is generally acceptable.

Note 4: the argument in exercise 11 may also be used to justify the premise in this question.

9. Show that the divergence of the energy-momentum tensor is zero, i.e. $T^{\mu\nu}_{\ \ ;\nu} = 0$.

Answer: If we accept that:

(**a**) the conservation of the momentum 4-vector (which is no more than a combination of the conservation of energy and the conservation of momentum 3-vector) is valid in the flat spacetime of special relativity (as well as classical mechanics) and

(**b**) these conservation principles should extend to the curved spacetime of general relativity (at least locally),

then the answer of the previous exercise should be sufficient to establish the stated premise, i.e. the divergence of the energy-momentum tensor is zero. In fact, (a) is a well established fact (see B4) while (b) is sensible and intuitive and hence the stated premise is justifiable (considering the issues indicated in the notes of the previous exercise such as local validity).[97]

10. Justify the existence of vacuum problems and vacuum solutions in general relativity, i.e. why we should have curved spacetime at locations where $T_{\mu\nu} = 0$.

Answer: The sensibility of the existence of vacuum problems and vacuum solutions in general relativity may seem conceptually problematic because they seem nonsensical, i.e. how a void region feels the effect of the presence of matter in another region if the equation is applied to the space point by point (since all tensors are coordinate-related) where the Field Equation in the vacuum region is supposed to become $0 = 0$ (since $T_{\mu\nu} = 0$ and $G_{\mu\nu} = 0$)?

The answer to this seeming paradox is that although the tensors in the Field Equation are coordinate-related and hence they apply to the spacetime point by point we are not solving this equation point by point but we are solving it analytically or numerically over the entire spacetime (or region of spacetime) and hence we correlate the geometry of the spacetime (or region of spacetime) as a whole and in its entirety (where this geometry is represented by the metric) to the physical setting of the spacetime as a whole (as represented by the energy-momentum tensor over the entire spacetime). More clearly, although we are looking for a solution in the vacuum region we actually consider the entire spacetime by considering the gravity source (which exists in the non-vacuum region) in the entire spacetime. In fact, if the vacuum problems and solutions in general relativity are nonsensical from this perspective then even the corresponding problems and solutions in classical gravity should be nonsensical because there is no difference between the formulation of general relativity and the field formulation of classical gravity from this perspective.

To be more clear, in the vacuum problems of general relativity we have $T_{\mu\nu} = 0$ in the vacuum region and this leads to $R_{\mu\nu} = 0$ (see exercise 7; also see exercise 5 of § 3.2). Now, $R_{\mu\nu}$ does not entirely represent the curvature (since the curvature is represented entirely by $R_{\mu\nu\sigma\omega}$) and hence its vanishing does not mean that the vacuum region is flat. However, $R_{\mu\nu}$ is entirely dependent on the metric of the spacetime and the metric at any location or region of the spacetime is dependent on $T_{\mu\nu}$ in the entire spacetime. Therefore, when we solve a vacuum problem we in fact take into consideration the metric (and hence the curvature) in the vacuum region where this (local) metric is determined (implicitly) by the energy-momentum tensor in the entire spacetime since the metric at any location or region is formed by the energy-momentum in the entire spacetime. In fact, this should provide more clarification about the issues discussed in exercise 7.

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[97] In the literature of general relativity, the significance of $T^{\mu\nu}_{\ \ ;\nu} = 0$ as a manifestation of energy-momentum conservation (like its counterpart $T^{\mu\nu}_{\ \ ,\nu} = 0$) is commonly restricted to flat spacetime where no gravity exists. So, in essence the equation $T^{\mu\nu}_{\ \ ;\nu} = 0$ is a generalization to the equation $T^{\mu\nu}_{\ \ ,\nu} = 0$ from coordinates perspective (or at most it is a manifestation of energy-momentum conservation excluding gravitational contributions).

Yes, the vacuum problems and vacuum solutions may be criticized from other perspectives such as ambiguity and partial representation of curvature information (see exercise 9 of § 2.13 and refer to § 10.1.15 and § 10.1.16).

11. What is the significance of the fact that the divergence of the energy-momentum tensor is zero? Also, link this to the vanishing of the divergence of the Einstein tensor.

Answer: Let first see what is the significance of the divergence of a rank-1 tensor (i.e. vector) field in 3D space. The divergence of a vector field is a measure for the spatial rate at which a certain quantity (e.g. mass) "diverges" or "converges" in the space. So, if we have a source or sink for that quantity in the concerned region of space then the divergence is non-vanishing because the quantity diverges or converges due to the positive or negative supply by the source or sink. Otherwise, the divergence should vanish because there is no supply. So, at the root of the divergence (or rather divergence-free) is the concept of conservation or the denial of creation and annihilation since each diverging quantity from a certain location in space should come from a source and each converging quantity to a certain location in space should go to some sink. On generalizing this idea and applying it to the energy-momentum tensor in 4D spacetime we can see that the vanishing of the divergence of the energy-momentum tensor originates from the conservation of energy and momentum, as we concluded in a rather different way earlier (see exercises 8 and 9).

The link of this to the vanishing of the divergence of the Einstein tensor should be obvious because the divergence-free nature of the Einstein tensor (see § 2.13) means that the curvature of spacetime is "conserved" in some sense and hence it can only be generated from a source of "energy-momentum". In fact, this link between the curvature of spacetime (as represented by the Einstein tensor) and the energy-momentum (as represented by the energy-momentum tensor) is embedded in the Field Equation which correlates the Einstein tensor to the energy-momentum tensor (and hence it correlates the divergence-free nature of the curvature to the divergence-free nature of the energy-momentum, as we will see in § 3).[98]

12. Make a comparison between the Einstein tensor **G** and the energy-momentum tensor **T**.

Answer: We note the following:

- Both **G** and **T** are rank-2 symmetric tensors in 4D spacetime and hence each has 16 components 10 of which are independent.
- The divergence of both tensors is zero.
- **G** and **T** are proportional to each other according to the Field Equation (see § 3.2).
- **G** is geometric while **T** is physical.
- The physical dimensions of **G** are reciprocal area while the physical dimensions of **T** are energy density.

13. Discuss briefly the historical origin of the energy-momentum tensor and if it is a novelty of general relativity.

Answer: According to the established historical records, the energy-momentum tensor of general relativity originates from the Minkowski (or Minkowski-Laue) energy tensor and hence it is not a novelty of the theory of general relativity although it was generalized and extended in this theory. This should add more credit to Minkowski for the development of the relativity theories (in addition to his fundamental contribution of the merge of space and time and the consequent formalism of 4D spacetime which is essential to general relativity in particular). We should also note that the energy-momentum tensor was used in some metric gravity theories that preceded general relativity such as the theory of Nordstrom (see § 1.2).

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[98] In fact, this is the essence of the general relativistic paradigm that the energy-momentum is the physical source of the geometric curvature which represents the gravity.

14. Comment on the following:
$R_{ij} = 0$ (as well as $R = 0$) globally does not imply flatness (globally) from a geometric perspective but it implies flatness (globally) from a general relativistic perspective (according to the Field Equation and as required by special relativity). However, $R_{ij} = 0$ (as well as $R = 0$) locally does not imply flatness (locally) from a geometric perspective and it does not imply flatness (locally) from a general relativistic perspective.
Answer: We note the following points:
• This should reflect the partial representation of curvature in general relativity.
• This may also explain the need for dark energy in general relativity and its applications.
• The difference between global and local in the implication of flatness and the applicability of special relativity requires consideration.

Chapter 3
Formalism of General Relativity

3.1 Rationale of the Field Equation

1. What we mean by the "Rationale of the Field Equation" which is the title of this section?
   Answer: We mean the basic theoretical framework which the fundamental formalism of general relativity (as represented by the Field Equation) is based on. In other words, the rationale represents the logic that underlies the development and application of the formalism of this theory as a gravity theory.
2. Outline the transition from classical gravity to general relativistic gravity.
   Answer:[99] In classical gravity we have gravitating mass and gravitated mass where these masses interact through the gravitational force field (which is created by the gravitating mass) in a flat 3D Euclidean "spatial" space. In general relativity the role of gravitating mass is taken over by the energy-momentum distribution which shapes the geometry of a curved 4D Riemannian spacetime resulting in free objects (whether massive or massless) being following the geodesic trajectories (or world lines) of this Riemannian spacetime and hence the role of gravitated mass becomes redundant. In fact, the redundancy of the gravitated mass reflects the spirit of the equivalence principle.[100]
3. In the literature of general relativity we can find several attempts and methods for deriving the Field Equation from more fundamental principles in a rigorous way. Discuss this issue.
   Answer: Yes, there are such attempts and methods (and hence our "Rationale" approach seems less rigorous and convincing). The most distinguished of these attempts and methods is the use of the variational methods and action principles. However, we should note that most (if not all) these attempts and methods either lack sufficient rigor or they are based on an "intrusive" insight (by anticipating the Field Equation in advance and hence they are not true derivations) or they are based on many approximations, assumptions and twitchy manipulations (among other cons). Accordingly, we are happy with our intuitive "Rationale" approach for justifying the foundations of the Field Equation and how it emerges (or how it is "derived").[101] The reality is that most physical theories are originally developed and emerge from such loose rationales and intuitive insights but they are justified later by alleged rigorous derivations. In fact, no derivation can validate a physical theory since the only true authority in physics is the experimental observations of Nature (unlike mathematics where derivation is the ultimate true authority). Yes, the derivation should rationalize the theory, demonstrate its interesting aspects and fit it in a more general picture and these factors (among other factors) should improve the understanding and appreciation of the theory and help to incorporate it within the existing science (which should play a positive role in the development of future science as well). In our view, the "Rationale" approach is as "rigorous" and valid as the other methods in achieving these objectives with minimum effort.

3.2 The Field Equation of General Relativity

1. How many component equations the tensorial Field Equation represents, and how many of these component equations are independent?
   Answer: The terms of the tensorial Field Equation are rank-2 symmetric tensors in 4D spacetime and hence the tensorial Field Equation represents 16 component equations 10 of which are independent.
2. What "solving the tensorial Field Equation" means mathematically and physically?
   Answer: Mathematically, it means solving a system of 10 simultaneous equations which represent its

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[99] In this answer we are considering the essence of classical gravity and general relativistic gravity as represented in a simple gravitational system made of a large gravitating object and a gravitated test object.

[100] As we will see, this in fact is an irony.

[101] In fact, our "Rationale" approach even deals with some aspects of applicability.

independent components. In some cases, some of these 10 equations are either trivial (e.g. $0 = 0$ or $\theta = \pi/2$) or they are identical or equivalent to other equations in the system and hence the number of independent and useful simultaneous equations of this system is reduced to less than 10 (see § 4). Physically, it means finding the metric tensor of the spacetime that corresponds to a given energy-momentum tensor which is determined according to the presumed physical setting. The determination of the metric will subsequently lead to the determination of the physical consequences of the metric such as the geodesic trajectories which are determined by using the geodesic equation which is based on the metric (see § 2.9.5).

3. What is the classical equation of gravity that corresponds to the tensorial Field Equation of general relativity? How many component equations the classical equation of gravity represents?
Answer: It is the Poisson gravitational field equation: $\nabla^2\Phi = 4\pi G\rho$ (or the equation $\nabla \cdot \mathbf{g} = -4\pi G\rho$ which is equivalent to the Poisson equation as explained in § 1.5). The classical equation of gravity is a scalar equation and hence it has a single component. This can be deduced from the fact that the Laplacian of a scalar is scalar (or the divergence of a vector is scalar) and can also be seen from inspecting the right hand side of the Poisson equation which is a product of scalars.
4. Show that $G = \kappa T$ and $R = -\kappa T$ where G, T, R are the traces of $G^\mu_\nu, T^\mu_\nu, R^\mu_\nu$.
Answer: We have:

$$\begin{aligned} G_{\mu\nu} &= \kappa T_{\mu\nu} \\ g^{\mu\nu}G_{\mu\nu} &= \kappa g^{\mu\nu}T_{\mu\nu} \\ G^\nu_\nu &= \kappa T^\nu_\nu \\ G &= \kappa T \\ -R &= \kappa T \\ R &= -\kappa T \end{aligned}$$

where in line 1 we use the Field Equation, in line 2 we multiply the equation with $g^{\mu\nu}$, in line 3 we contract the indices, in line 4 we use the definition of G and T as traces of G^μ_ν and T^μ_ν, and in line 5 we use $G = -R$ which we proved in exercise 6 of § 2.13.

5. Show that $R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)$.
Answer: We have:

$$\begin{aligned} G_{\mu\nu} &= \kappa T_{\mu\nu} \\ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= \kappa T_{\mu\nu} \\ R_{\mu\nu} &= \kappa T_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R \\ R_{\mu\nu} &= \kappa T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\kappa T \\ R_{\mu\nu} &= \kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \end{aligned}$$

where in line 1 we use the Field Equation, in line 2 we use the definition of Einstein tensor, and in line 4 we use $R = -\kappa T$ which we proved in the previous exercise.

6. Show that if a region of spacetime is void of matter and energy then the Field Equation in that region becomes: $R_{\mu\nu} = 0$. What is the significance of this?
Answer: From the previous exercise we have:

$$R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \tag{16}$$

Now, if a region of spacetime is void of matter and energy then the energy-momentum tensor $T_{\mu\nu}$ in that region is zero and hence its trace T must also be zero. Accordingly, the above equation becomes

$R_{\mu\nu} = 0$ in the void region.

The significance of this is that the Field Equation has vacuum solutions, i.e. solutions in void regions of spacetime where $R_{\mu\nu} = 0$ applies.

Note 1: the relation $R_{\mu\nu} = 0$ applies in flat spacetime as well as in void regions of curved spacetime and hence from this perspective (as applied in a certain region of spacetime) it is ambiguous with regard to the flatness and curvedness of the spacetime. The distinction between the two cases is that in flat spacetime this relation applies over the entire spacetime while in curved spacetime it applies only in certain regions (i.e. void regions). This is because if the entire spacetime is void of matter and energy then in any region of this spacetime we must have $R_{\mu\nu} = 0$ (i.e. the Ricci curvature tensor vanishes identically throughout the space). This should be obvious because if the spacetime is void of matter and energy then it is flat and hence its Riemann-Christoffel curvature tensor is zero identically[102] and therefore its Ricci curvature tensor is also zero identically. On the other hand, if only certain regions of spacetime are void then there should be some regions in spacetime that are not void and hence the spacetime contains some matter or/and energy and therefore it should be curved. Accordingly, we will have $R_{\mu\nu} = 0$ in the void regions although the spacetime is curved.

Note 2: the issue in the previous note is related to the issue that we discussed earlier that is the provision $R_{\mu\nu} = 0$ identically is a necessary but not sufficient condition for space flatness. What is necessary and sufficient condition for space flatness is the vanishing of the Riemann-Christoffel curvature tensor $R_{\mu\nu\sigma\rho}$ identically (see § 2.10), and we saw earlier (refer to § 2.11) that the vanishing of the Ricci curvature tensor identically does not imply the vanishing of the Riemann-Christoffel curvature tensor identically. In fact, our discussion earlier was from a purely mathematical and geometric perspective. Now, from a physical perspective where $R_{\mu\nu}$ is linked to $T_{\mu\nu}$ through the Field Equation the situation has changed because the provision $R_{\mu\nu} = 0$ is now a necessary and sufficient condition for spacetime flatness when this provision applies identically over the entire spacetime[103] and hence it is like the provision $R_{\mu\nu\sigma\rho} = 0$. This can be explained by our assertion that the gravity-related curvature of spacetime is a particular part of the entire curvature of the spacetime and hence even if the "geometric" curvature of spacetime did not vanish identically (i.e. $R_{\mu\nu\sigma\rho} \neq 0$ identically) the "physical" curvature (i.e. the curvature that represents gravity according to general relativity) could still vanish because it is just a part of the entire curvature. In fact, this may be linked to the issues of gravity of gravity, non-linearity and the conservation of energy-momentum in the spacetime of general relativity as well as the issue of $T_{\mu\nu}$ being partially representing the source of gravity.[104] However, these issues are messy and lengthy to discuss and hence we only draw the attention of the reader to the significance of these issues in the present context without going through detailed investigation and discussion (although some of these issues will be dealt with rather briefly later on).

7. Does the condition $R_{\mu\nu} = 0$ at a given region of spacetime means that the given region is flat? Justify your answer.

   Answer: The condition $R_{\mu\nu} = 0$ at a given region of spacetime implies that $T_{\mu\nu} = 0$ at that region (see for example exercise 9) but it does not imply $T_{\mu\nu} = 0$ at every region in the spacetime. So the answer is No, because the given region could be in the neighborhood of a region that is not void of matter and energy. In other words, a spacetime is flat if the entire spacetime is void of matter and energy.[105] Yes, in a region that is void of matter and energy and it is very far from aggregates of matter and energy the spacetime is effectively flat. So in brief, flatness does not necessarily apply to a region of spacetime if the region is void of matter and energy because the void region could be in a spacetime which is not void of matter and energy (i.e. the void region could be in the neighborhood of a region that is not void).

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[102] This is based on assuming (arguably) that the entire curvature of spacetime originates from the energy-momentum (see exercise 9 of § 2.13). This may be addressed by saying: "if the spacetime is void of matter and energy then the gravity-related part of its curvature is zero" although this will not necessarily lead to the required conclusion. In fact, if we have to be more precise then the above statement should read: "because if the spacetime is void of matter and energy then (according to general relativity) it is flat and hence (according to differential geometry) its Riemann-Christoffel curvature tensor is zero identically". As we see (and will see further) this is not entirely consistent. Anyway, the above proposition (i.e. global vanishing of matter and energy leads to global vanishing of $R_{\mu\nu}$) can be obtained directly from Eq. 16 although this on its own should not imply global flatness.

[103] This can be concluded from the result of exercise 9, i.e. $R_{\mu\nu} = 0$ *iff* $T_{\mu\nu} = 0$ (when applied over the entire spacetime accompanied with the "possibly arguable" claim that $T_{\mu\nu} = 0$ identically is a necessary and sufficient condition for spacetime flatness).

[104] In fact, these issues should also be linked to the issue of cosmological constant and dark energy (although this is a cosmological, rather than physical, issue).

8. Show that $G_{\mu\nu} = 0$ *iff* $T_{\mu\nu} = 0$ (where these conditions apply locally or globally).
Answer: According to the Field Equation we have $G_{\mu\nu} = \kappa T_{\mu\nu}$. Hence:
If $G_{\mu\nu} = 0$ then $T_{\mu\nu} = 0$ since $\kappa \neq 0$.
If $T_{\mu\nu} = 0$ then $G_{\mu\nu} = 0$.
9. Show that $R_{\mu\nu} = 0$ *iff* $T_{\mu\nu} = 0$ (where these conditions apply locally or globally).
Answer: We have two parts:
(**a**) If $R_{\mu\nu} = 0$ then $T_{\mu\nu} = 0$: in this case we have:

$$\begin{aligned} R_{\mu\nu} &= \kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \\ T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T &= 0 \\ g^{\mu\nu}T_{\mu\nu} - \frac{1}{2}g^{\mu\nu}g_{\mu\nu}T &= 0 \\ T^{\nu}_{\nu} - \frac{1}{2}\delta^{\nu}_{\nu}T &= 0 \\ T - 2T &= 0 \\ T &= 0 \end{aligned}$$

where line 1 is from exercise 5, line 2 is from the condition $R_{\mu\nu} = 0$ (noting that $\kappa \neq 0$), in line 3 we multiply with $g^{\mu\nu}$, and in lines 4 and 5 we contract the indices and use the definition of trace plus the inverse relation between the covariant and contravariant metric tensor and $\delta^{\nu}_{\nu} = 4$. Hence:

$$\begin{aligned} R_{\mu\nu} &= \kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \\ T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T &= 0 \\ T_{\mu\nu} &= 0 \end{aligned}$$

(**b**) If $T_{\mu\nu} = 0$ then $R_{\mu\nu} = 0$: in this case we have $T = 0$ (since $T_{\mu\nu} = 0$) and hence from the relation $R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)$ which we obtained in exercise 5 we get $R_{\mu\nu} = 0$.
Note: a shorter answer is:
If $R_{\mu\nu} = 0$ then $G_{\mu\nu} = 0$ (according to exercise 7 of § 2.13) and from the Field Equation $G_{\mu\nu} = \kappa T_{\mu\nu}$ we get $T_{\mu\nu} = 0$ since $\kappa \neq 0$.
If $T_{\mu\nu} = 0$ then $G_{\mu\nu} = 0$ (according to the Field Equation) and hence $R_{\mu\nu} = 0$ (according to exercise 7 of § 2.13).
10. Compare the relations that we derived in § 2.13 (e.g. $G = -R$) with the relations that we obtained in this section, e.g. $R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)$.
Answer: The relations that we derived in § 2.13 are based on the definition of **G** (aided by mathematical relations and identities from differential geometry) and hence they are fundamental relations, while the relations that we obtained in this section are based on the Field Equation (possibly with the help of some fundamental relations such as those derived in § 2.13) and hence their validity is dependent on the validity of the Field Equation. In other words, the relations in § 2.13 are purely geometric relations based on the definitions and rules of differential geometry while the relations in this section are geometric-physical relations based on general relativity and its Field Equation. In fact, differential geometry itself cannot determine the relation between $T_{\mu\nu}$ and any geometric tensor like

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[105] We repeat our reservation that we expressed in footnote [103].

$R_{\mu\nu}$ because such relations are based on the validity of general relativity and its Field Equation which contains a physical (i.e. non-geometric) element and hence differential geometry on its own cannot determine such correlations between these tensors (i.e. geometric tensors and physical tensors). To sum up, the relations that we derived in § 2.13 are mathematical identities while the relations that we obtained in this section are supposed to be physical laws.

11. Can you see a potential problem in the Field Equation from the perspective of the fact that the Einstein tensor contains less curvature information than the Riemann-Christoffel curvature tensor?
Answer: In a spacetime whose Riemann-Christoffel curvature tensor is not zero (and hence the spacetime is curved) but its Ricci curvature tensor (and hence its Einstein tensor) is zero we have $T_{\mu\nu} = 0$ (according to the Field Equation; see exercises 8 and 9) and therefore the spacetime is curved without the presence of matter and energy. This should be a consequence of the fact that less curvature information is contained in the Ricci curvature tensor (and hence in the Einstein tensor) than the curvature information contained in the Riemann-Christoffel curvature tensor (see exercise 4 of § 2.12 and exercise 9 of § 2.13). Accordingly, the gravity is related to a given part of the spacetime curvature (i.e. the part represented by the Ricci curvature tensor which is linked to the energy-momentum tensor) but not to all parts of the spacetime curvature since the part of the curvature that is represented solely by the Riemann-Christoffel curvature tensor (i.e. excluding the part represented by the Ricci curvature tensor) is not related to the energy-momentum tensor and hence it does not represent gravity. A theoretical justification for the exceptional status of the "gravity-privileged" part of the spacetime curvature and why the gravity should depend specifically on this part but not on the other part should be provided by general relativity. Some aspects of this issue have been investigated earlier (see for example exercise 6) and will be investigated further later on.
12. Discuss the principle of geodesic motion in general relativity and its relation to the Field Equation.
Answer: This principle (which is one of the pillars of general relativity and was mentioned in § 1.8.4) states that free objects follow geodesic trajectories (or world lines) in the spacetime. It is shown in the literature of general relativity that this principle can be obtained (although in a rather twitchy way) from the Field Equation and hence we do not need to postulate this principle independently.
13. Analyze the significance of the vanishing of the energy-momentum tensor locally and globally within the context of the Field Equation.
Answer: In exercise 9 it was shown that $R_{\mu\nu} = 0$ *iff* $T_{\mu\nu} = 0$. Accordingly, we can make the following points:
• $T_{\mu\nu} = 0$ globally means $R_{\mu\nu} = 0$ globally, i.e. the Minkowski spacetime of special relativity can be seen as a special solution to the Field Equation represented by $R_{\mu\nu} = 0$.[106] Similarly, $T_{\mu\nu} = 0$ locally means $R_{\mu\nu} = 0$ locally which is the vacuum equation of the curved spacetime of general relativity. In brief, $R_{\mu\nu} = 0$ represents both the Minkowski spacetime of special relativity and the vacuum region of spacetime of general relativity. We may also say rather differently: the vacuum equation $R_{\mu\nu} = 0$ represents special relativity as well as general relativity in the vacuum region of spacetime. Accordingly, we may claim that there is an ambiguity in $R_{\mu\nu} = 0$ from this aspect.
• The general relativistic premise that $R_{\mu\nu} = 0$ *iff* $T_{\mu\nu} = 0$ (which is obtained directly from the Field Equation) implies that part of the curvature of spacetime is not represented by the Field Equation because from a purely geometric perspective (i.e. according to differential geometry) the equation $R_{\mu\nu} = 0$ does not imply the flatness of spacetime since $R_{\mu\nu} = 0$ (globally) is a necessary but not sufficient condition for space flatness (refer to exercise 5 of § 2.11) while from a general relativistic perspective the equation $R_{\mu\nu} = 0$ (globally) is a necessary and sufficient condition for space flatness since $T_{\mu\nu} = 0$ (globally) is a necessary and sufficient condition for space flatness according to general relativity (or at least that is what it should be according to its principles).
• The vacuum equation $R_{\mu\nu} = 0$ of general relativity (which we obtain from $T_{\mu\nu} = 0$) cannot be solved in general unless we start from a guess about the metric. Hence, it cannot be solved by the natural (or ideal) method of obtaining the metric from the mass-energy distribution (as explained in § 4) and this should introduce difficulties in solving the vacuum equation in most cases (refer to 10.1.16 for more

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[106] We are not considering the cosmological constant and the dark energy, or if they are considered their effect is incorporated in $T_{\mu\nu}$ (see § 3.3).

details).

14. Consider obtaining the Field Equation as a generalization of the classical gravity with the help of special relativity and some general principles (e.g. principle of invariance, principle of equivalence, principle of metric gravity, etc.).

Answer: Although general relativity (as represented mainly by the Field Equation) is commonly depicted as a "miraculous achievement" it can be obtained rather straightforwardly as a generalization of the fundamental idea of classical gravity with the presumption of the restricted validity of special relativity plus some general epistemological and physical principles. In the following points we outline the main elements of the argument that naturally leads to general relativity and the derivation of the Field Equation:

- We are supposed to construct a field theory of gravity based on the paradigm of 4D spacetime whose origin comes from special relativity. Therefore, it is natural to start from the field formulation (rather than force formulation) of classical gravity as represented by the Poisson equation: $\nabla^2\Phi = 4\pi G\rho$. Now, the source of gravity in the Poisson equation is the mass density ρ. However, in special relativity mass and energy are equivalent (according to the Poincare relation $E = mc^2$) and hence mass is a special (i.e. "condensed") form of energy. Therefore, a natural generalization of this classical source of gravity is energy density (which can be represented generically by ρc^2).
- To achieve form invariance of physical laws (as required by the invariance principle) we need to adopt a tensorial approach in the formulation of gravity. Moreover, because we are supposed to build on the methods and results of special relativity a 4D spacetime multi-dimensional approach should be employed to underlie the new gravity theory. This means that gravity is fundamentally a tempo-spatial phenomenon (rater than a purely spatial phenomenon as in classical gravity). It also means that the new formulation should be 4D tensorial rather than scalar. In other words, to build an invariant tensorial gravity equation in 4D spacetime the new formulation should have more than one component, i.e. it should not be scalar like the field formulation of classical gravity.
- Now, in classical gravity energy is a scalar quantity (and this may partly explain why the classical formulation is scalar) but in special relativity (which the new theory is supposed to build on) energy is the (scaled) temporal component of the momentum 4-vector. So, a 4D tensorial generalization of classical gravity should use the 4-momentum to replace the classical energy. In other words, the source of gravity in the new theory is the "4-momentum flow density" (whose all components have the physical dimensions of energy density as seen in § 2.14). In fact, our definition of the energy-momentum tensor in § 2.14 as the flux of the momentum 4-vector meets this criterion since from a simple dimensional analysis a tempo-spatial tensor in 4D spacetime that generalizes the scalar "energy density" of classical gravity through the use of the rank-1 4-momentum should be a rank-2 tensor (i.e. $T^{\mu\nu}$) representing the flux of the 4-momentum. As explained earlier (see § 2.14) the special relativistic demand $T^{\mu\nu}_{\ ,\nu} = 0$ that is based on the conservation of 4-momentum should be generalized (covariant differentially) to $T^{\mu\nu}_{\ ;\nu} = 0$, i.e. the tensor $T^{\mu\nu}$ that represents the source of gravity in the new theory should be divergence-free.
- We generalized so far the source of gravity (i.e. ρ) on the right hand side of the Poisson equation of classical gravity by replacing it with $T^{\mu\nu}$. So, what we need now is to generalize the left hand side of the Poisson equation to match our generalization on the right hand side. In other words, we need a rank-2 tensorial formulation that represents a second order derivative of a sort of "potential" (to be like $\nabla^2\Phi$ in the Poisson equation).
- If we recall the fundamental idea of general relativity that deems gravity as a curvature of spacetime created by the source of gravity (which we called "the principle of metric gravity") then the left hand side should represent this curvature. Now, if we inspect the mathematical expressions of the various types of curvature[107] we note that curvature is essentially a second order derivative of the metric and hence the metric should stand for the gravitational potential in the new theory.

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[107] For example, the Riemann-Christoffel curvature is given by:

$$R_{ijkl} = \frac{1}{2}\left(\partial_k\partial_j g_{li} + \partial_l\partial_i g_{jk} - \partial_k\partial_i g_{jl} - \partial_l\partial_j g_{ki}\right) + [il, r]\,\Gamma^r_{jk} - [ik, r]\,\Gamma^r_{jl}$$

• We may also pose the argument in the previous point rather differently by saying: if gravity should be represented by the geometry of the spacetime then the metric (which represents the geometry) should correspond to the classical gravitational potential Φ in the Poisson equation. Therefore, to generalize the left hand side of the Poisson equation the left hand side in the new theory should stand for the curvature (which is essentially a second order derivative of the metric and hence it is like $\nabla^2\Phi$).
• So, to match the new source of gravity (as represented on the right hand side by $T^{\mu\nu}$ which is a rank-2 symmetric divergence-free tensor) the left hand side should be represented by a tensor of similar properties (i.e. rank-2 symmetric divergence-free). Moreover, since the curvature of space is primarily represented by the Riemann-Christoffel curvature tensor (which unfortunately cannot be used here because it is rank-4) the required curvature tensor should be derived from the Riemann-Christoffel curvature tensor (through contraction which reduces the rank from 4 to 2). So, a natural choice is the Ricci curvature tensor $R^{\mu\nu}$ (which seems a reasonable choice since in special relativity $T^{\mu\nu} = 0$, because $T^{\mu\nu}$ as a source of gravity should vanish, and hence $R^{\mu\nu} = 0$). However, although $R^{\mu\nu}$ is rank-2 symmetric it is not divergence-free and hence it is not a good match to $T^{\mu\nu}$. With a few trials, we can find that the combination $R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$ (which is a rank-2 symmetric tensor that is derived from the Riemann-Christoffel curvature tensor by contractions and is totally based on the metric) is divergence-free (as well as rank-2 symmetric) and hence it is a good match to $T^{\mu\nu}$. Accordingly, we can safely assume that the obtained source and curvature tensors are proportional and hence the new tensorial formulation of gravity should take the following form:

$$\begin{aligned}
R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R &= \kappa T^{\mu\nu} \\
G^{\mu\nu} &= \kappa T^{\mu\nu} \\
G^{\mu\nu} &= -\frac{8\pi G}{c^4}T^{\mu\nu}
\end{aligned}$$

where κ is the proportionality constant, $G^{\mu\nu}$ (which symbolizes $R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$) is the Einstein tensor, and the equality $\kappa = -\frac{8\pi G}{c^4}$ is obtained from the classical limit and supported by dimensional analysis (refer to § 5.1) with G representing the gravitational constant and c the speed of light.
Note 1: in anticipation of the next section (and possibly for other legitimate purposes), we may adjust the last point slightly by saying: since $T^{\mu\nu}$ and $G^{\mu\nu}$ are divergence-free they are linked by a linear relationship (rather than a proportionality) and hence we should also have a constant term which stands for the cosmological term (and can possibly stand for other things).
Note 2: the rationale in our argument in this exercise will be more clear if we label the momentum 4-vector as the "energy-momentum vector". However, we avoided this to avoid potential confusion with the energy-momentum tensor although they are clearly distinguished by "vector" and "tensor".
Note 3: although "4-vector" and "4-tensor" (and their alike) may be associated conventionally with the spacetime of special relativity (see § 2.2), they should be used here in a more general sense (i.e. as generalization from their special relativistic sense).

3.3 The Field Equation with Cosmological Constant

1. Using the Field Equation with the cosmological term, show that $G = \kappa T - 4\Lambda$ and $R = -\kappa T + 4\Lambda$.
Answer: We have:

$$\begin{aligned}
G_{\mu\nu} &= \kappa T_{\mu\nu} - \Lambda g_{\mu\nu} \\
g^{\mu\nu}G_{\mu\nu} &= \kappa g^{\mu\nu}T_{\mu\nu} - \Lambda g^{\mu\nu}g_{\mu\nu} \\
G^{\nu}_{\nu} &= \kappa T^{\nu}_{\nu} - \Lambda\delta^{\nu}_{\nu} \\
G &= \kappa T - 4\Lambda \\
-R &= \kappa T - 4\Lambda \\
R &= -\kappa T + 4\Lambda
\end{aligned}$$

where in line 1 we use the Field Equation with the cosmological term, in line 2 we multiply the equation with $g^{\mu\nu}$, in line 3 we contract the indices and use $g^{\nu}_{\nu} \equiv \delta^{\nu}_{\nu}$, in line 4 we use the definition of G and T as traces of G^{μ}_{ν} and T^{μ}_{ν} and $\delta^{\nu}_{\nu} = \delta^0_0 + \delta^1_1 + \delta^2_2 + \delta^3_3 = 4$, and in line 5 we use $G = -R$ which we proved previously (see exercise 6 of § 2.13).

2. Show that $R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) + \Lambda g_{\mu\nu}$ and comment on its significance.
Answer: From the Field Equation (with the cosmological term) we have:

$$\begin{aligned}
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= \kappa T_{\mu\nu} - \Lambda g_{\mu\nu} \\
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\left(-\kappa T + 4\Lambda\right) &= \kappa T_{\mu\nu} - \Lambda g_{\mu\nu} \\
R_{\mu\nu} + \frac{1}{2}\kappa T g_{\mu\nu} - 2\Lambda g_{\mu\nu} &= \kappa T_{\mu\nu} - \Lambda g_{\mu\nu} \\
R_{\mu\nu} &= \kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) + \Lambda g_{\mu\nu}
\end{aligned}$$

where in line 2 we use $R = -\kappa T + 4\Lambda$ (which we obtained in the previous exercise).
Comment: if the energy-momentum tensor is zero we get $R_{\mu\nu} = \Lambda g_{\mu\nu}$ which means that the Minkowski spacetime of special relativity is not a solution unless $\Lambda = 0$ (considering the global validity of these equations and conditions). Yes, because the cosmological constant is supposedly very tiny (i.e. $\Lambda \simeq 0$) the spacetime is approximately flat Minkowskian.

3. What is the relation between the cosmological term and dark energy?
Answer: The cosmological term (as a physical factor) is supposed to represent the contribution of dark energy to the physical source of gravity which is the distribution and flow of matter and energy as represented primarily by the energy-momentum tensor.

4. Discuss dark energy and assess it as a physical or metaphysical concept. Also, discuss and assess dark matter from the same perspective.
Answer: Physical concepts are obtained from direct or indirect observations of the Universe and are supported by experimental evidence, while metaphysical concepts are arbitrary and *ad hoc* concepts and they have no experimental evidence or justification. The concept of dark energy is proposed and used in modern physics to address certain limitations and shortcomings of the existing gravitational theories and models (especially in cosmology) which failed to provide logical consequences and comply with observations.
Regarding dark matter, it is physically different from dark energy although in most of its instances it is similar to dark energy in being a metaphysical entity. In brief, to keep the aforementioned theories and models alive and rectify their failures to provide reliable predictions it is proposed that there is a "dark" or "unseen" amount of matter in the Universe whose existence is necessary to make these theories and models logical and functioning and compensate for their failure.
Accordingly, both dark energy and dark matter are improvised unobservable entities that were created to fix the failure of our theories and hence they do not belong to physics which is an experimental science. In fact, these concepts resemble the concepts of "ghost" and "angel" in their relation to science.
Note: we should distinguish between two types of dark matter: ordinary dark matter that is "dark" because it cannot be observed directly (e.g. matter of black holes or cold gas and dust in the interstellar medium) and non-ordinary dark matter that is entirely hypothetical and based on pure theoretical models and rationales (e.g. non-baryonic matter in some cosmological models or matter whose existence cannot be established independently of the theory). Although we can accept the former as physical (when there is sufficient indirect evidence on its existence but not when it is an *ad hoc* fix), we should reject the latter as metaphysical.

5. Outline the characteristics of the cosmological term $\Lambda g_{\mu\nu}$.
Answer: We note the following:
• The cosmological term is a modification to the Field Equation that was introduced originally to make the Universe static when it was noted that some cosmological models and metrics lead to dynamic (i.e. expanding or shrinking) Universe. However, it is currently used by relativistic cosmologists even with

non-static Universe.
• The cosmological term is a rank-2 symmetric tensor since it is a scalar Λ times a rank-2 symmetric tensor $g_{\mu\nu}$. Moreover, it has identically vanishing divergence; i.e. $(\Lambda g^{\mu\nu})_{;\nu} = \Lambda g^{\mu\nu}_{;\nu} = 0$ since the covariant derivative of the metric tensor is zero (see § 2.7). Also, the term has the physical dimensions of reciprocal area. Hence, the cosmological term has the same properties as the other terms of the Field Equation and therefore it is consistent with the other terms.
• Unlike the other geometric terms in the Filed Equation, the cosmological term does not vanish (assuming $\Lambda \neq 0$) when the mass-energy source of gravity vanishes[108] and this has important cosmological consequences (according to relativistic cosmology).[109]

6. Outline the characteristics of the cosmological constant Λ.
   Answer: We note the following:
   • It is a scalar that can be positive or negative or zero.
   • It has the physical dimensions of curvature (i.e. reciprocal area).
   • The name "cosmological constant" originates from its cosmological root (i.e. to keep the Cosmos static). Moreover, it is too small (assuming it is not null) that its effect becomes evident only on a cosmological scale (and hence it is relevant to be considered only in cosmological investigations).
   • It may be seen as a quantitative indicator of the "dark energy density".
   • Its sign indicates the type of gravitational effect of the dark energy, i.e. $\Lambda > 0$ means repulsive gravitational effect while $\Lambda < 0$ means attractive gravitational effect and hence it is a strange form of "gravity source". In fact, the strange nature of this improvised cosmological constant and what it represents should add more credibility to our assertion that it is a metaphysical (or rather mythical) object.
7. Discuss briefly some nonsensical consequences and implications that are commonly associated with the cosmological term and constant.
   Answer: We find in this context a number of counter-intuitive and non-physical things like negative pressure, negative gravity and anti-gravity. In fact, the features of general relativity that allow for example (through the cosmological constant and its alike) positive pressure to slow down the expansion of the Universe (or things like negative gravity or anti-gravity) should be considered as some of the nonsensical results and consequences of general relativity and its Field Equation because such strange things are not observed in any physical system and hence they should be regarded non-physical (like negative energy and imaginary roots) and therefore they must be rejected. The theory that leads to such nonsensical results and consequences should also be rejected or rectified.
8. Discuss the implication of the presumed existence of dark energy on special relativity.
   Answer: The presumed existence of dark energy should invalidate the global application of special relativity altogether because there will be no Minkowski spacetime since spacetime (even if it is void of ordinary sources of gravity) should have an inherent curvature due to the existence of this exceptional source of gravity. In fact, this applies even if the cosmological term represents a geometric factor since it then reflects inherent curvature of spacetime and hence the flat Minkowski spacetime does not exist even in the absence of all ordinary sources of gravity. However, the alleged tininess of the cosmological constant (and hence possible insignificance of dark energy density) could justify the global application of special relativity as an approximation.

3.4 The Linearized Field Equation

1. The linearization process is solely related to the metric and the geometric entities that are based on the metric. Why?
   Answer: Because the spacetime in the linearized form is assumed quasi-Minkowskian, as expressed

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[108] It should be obvious that we are treating the cosmological term as a geometric factor.

[109] In fact, this should be linked to the issue that we discussed earlier that is in general relativity the gravity (as sourced by the energy-momentum tensor) is correlated to only part of the curvature of spacetime (as represented by the Riemann-Christoffel curvature tensor) and this may explain the need for dark energy in cosmology to compensate for this partial representation. In other words, if the source of gravity is correlated to the entire curvature of spacetime, the dark energy could become redundant (because there is no missing part of curvature that requires dark energy to explain).

in the metric $g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon_{\mu\nu}$. Accordingly, the linearization is related to the geometry of spacetime, and the geometry is represented by the metric tensor and its companions (i.e. curvature tensors).

2. Show that $g^{\mu\nu} = \eta^{\mu\nu} - \varepsilon^{\mu\nu}$.
   Answer: The contravariant metric tensor $g^{\mu\nu}$ is the inverse of the covariant metric tensor $g_{\mu\nu}$ (i.e. $g_{\mu\alpha}g^{\alpha\nu} = \delta_\mu^\nu$) and hence if in the first order approximation $g^{\mu\nu} = \eta^{\mu\nu} - \varepsilon^{\mu\nu}$ then we should have $g_{\mu\alpha}g^{\alpha\nu} = (\eta_{\mu\alpha} + \varepsilon_{\mu\alpha})(\eta^{\alpha\nu} - \varepsilon^{\alpha\nu}) = \delta_\mu^\nu$ in this order of approximation. This can be easily shown as follows:
$$\begin{aligned} g_{\mu\alpha}g^{\alpha\nu} &= (\eta_{\mu\alpha} + \varepsilon_{\mu\alpha})(\eta^{\alpha\nu} - \varepsilon^{\alpha\nu}) \\ &= \eta_{\mu\alpha}\eta^{\alpha\nu} - \eta_{\mu\alpha}\varepsilon^{\alpha\nu} + \varepsilon_{\mu\alpha}\eta^{\alpha\nu} - \varepsilon_{\mu\alpha}\varepsilon^{\alpha\nu} \\ &= \eta_{\mu\alpha}\eta^{\alpha\nu} - \eta_{\mu\alpha}\varepsilon^{\alpha\nu} + \varepsilon_{\mu\alpha}\eta^{\alpha\nu} \\ &= \eta_{\mu\alpha}\eta^{\alpha\nu} - \varepsilon_\mu^{\ \nu} + \varepsilon_\mu^{\ \nu} \\ &= \eta_{\mu\alpha}\eta^{\alpha\nu} \\ &= \delta_\mu^\nu \end{aligned}$$
   where in line 3 we discard the term $\varepsilon_{\mu\alpha}\varepsilon^{\alpha\nu}$ because it is of second order, in line 4 we use $\eta_{\mu\alpha}$ and $\eta^{\alpha\nu}$ as index lowering and index raising operators, and in line 6 we use the reciprocity relation in the Minkowski metric.
3. Assess the reliability of the linearized form of the Field Equation.
   Answer: Since the linearized form is based on many approximations and simplifications (some of which are outlined in our derivation), its reliability could be questioned in many cases and circumstances[110] and hence any solution or formulation obtained from the linearized form should be treated with caution. This is particularly true when exact solutions and formulations are unknown and hence the results cannot be assessed independently. An example of this is gravitational waves where the commonly accepted formulation is based on the employment of the linearized form with no available non-linearized formulation to check (see § 8.6 and § 9.6).[111] In brief, the non-linearity of the Field Equation reflects important aspects of gravitation and these aspects are lost by linearization.[112] Therefore, the use of the linearized form should be closely monitored and checked.
4. What is the significance of linearizing the Field Equation?
   Answer: According to general relativists (or at least some of them) the non-linearity of the Field Equation represents the contribution of gravity itself to gravitation (i.e. the contribution of the gravitational field itself as a source of energy-momentum to the gravitational field). Accordingly, linearizing the Field Equation means removing this source of gravitation from the general relativistic formulation of gravity.
   Note: in fact, we believe that this "non-linear" aspect of the formalism of general relativity (as represented by the Field Equation) is a form of circularity (see § 10.1.14) or infinite sequence (rather than a form of non-linearity although it should lead to non-linearity) since "gravity gravitates" according to general relativists. The nonsensical aspect of this can be revealed by restating this as: "gravity gravitates itself" or "gravity gravitates and this gravity gravitates and this gravity gravitates ... etc.".

3.5 General Relativity as Gravity Theory and as General Theory

1. Outline the essence of the theory of general relativity as a "General Theory" that has gravitational and

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[110] For example, it is shown in the literature that the linearized form of the Field Equation leads to straight geodesic world lines of test objects, i.e. annulling the effect of gravity in this case.

[111] Although there are claims in the recent research literature of non-linear solutions to the gravitational waves problem, among the many controversies and ambiguities that surround this issue no definite and commonly accepted non-linearized formulation of the gravitational waves does exist and hence the generally accepted gravitational waves theory is based on the linearized form of general relativity. It should be noted that the theoretical confirmation of the existence of non-linear solutions (whose proofs are claimed in the literature) does not mean the actual existence (i.e. actual availability) but it means that such solutions can be derived.

[112] According to the literature, the non-linearity represents the gravity of gravity (or at least this is one important aspect of non-linearity).

non-gravitational formalisms.
Answer: The essence of general relativity as a "General Theory" can be summarized in the following points:

• We start from special relativity whose domain is inertial frames in a flat 4D Minkowski spacetime where the Lorentz transformations are the valid coordinate transformations across these frames. So, we have a 4D flat spacetime coordinated by global inertial frames in which the rules of special relativity apply globally.

• The theory of general relativity is seen as an extension and generalization to special relativity and hence the domain of special relativity should be extended to include all types of frame, i.e. non-inertial as well as inertial.

• Instead of dealing with gravity as a force in the spacetime and hence it is a physical phenomenon that takes place in spacetime and is observed in inertial and non-inertial frames, gravity is dealt with as a spacetime attribute by modeling it as a curvature of spacetime. By this approach, gravity is converted from being a physical phenomenon to be a geometric attribute and hence it should be dealt with as a spacetime and frame issue. Accordingly, a new type of frame (i.e. gravitational frame which means a frame of spacetime with gravity) is introduced and hence we have three main types of frame: inertial, accelerating and gravitational.

• By the weak equivalence principle, gravitational frames (which are introduced to deal with gravity) are included in accelerating frames (or the other way around) and hence we have two "main" types of frame: inertial and accelerating (or inertial and gravitational). Accordingly, "frames" in general relativity includes all types of coordinate systems of the 4D spacetime, whether inertial or accelerating and whether in the presence of gravity or in the absence of gravity (i.e. whether the spacetime is curved or flat).

• Also by the equivalence principle, freely falling frames in the presence of gravity are classified as inertial since the effect of free fall acceleration is canceled by the opposite effect of gravity. Hence, these frames are essentially special relativistic frames. To cancel any effect of curvature due to gravity (since gravity is modeled as curvature of spacetime) the locality condition is imposed so that the spacetime in freely falling frames is essentially flat. The imposition of the locality condition is facilitated by the adoption of the Riemannian geometry and Riemannian space which are necessarily flat at local level.

• We then have two independent formalisms in general relativity as a "General Theory": a gravitational formalism represented by the Field Equation and a non-gravitational formalism represented by special relativity. Now, since gravity is considered a geometric attribute we can say that in general relativity we have a geometric formalism represented by the Field Equation and a physical formalism represented by the rules of special relativity. These two formalisms are independent and hence the correctness of one of these formalisms does not guarantee the correctness of the other formalism although they must be consistent with each other if they should be correct.

• With regard to the validation of these formalisms, the gravitational formalism relies on gravitational evidence (e.g. planetary motion) which are generally based on the weak equivalence principle, while the non-gravitational formalism relies on non-gravitational evidence (e.g. the validity of the laws of electromagnetism in accelerating and gravitating frames) which are generally based on the strong equivalence principle. We should remark that the evidence for the validity of special relativity within its domain of inertial frames cannot be regarded as evidence for the validity of special relativity within the domain of general relativity (i.e. frames of reference in general including accelerating and gravitating frames even with the consideration of free fall and locality).

• Finally, we would like to emphasize on the importance of the strong equivalence principle for the validity of general relativity as a "General Theory" and the need for independent evidence for the validity and general applicability of this principle even if the gravitational component of the formalism of general relativity is proved to be correct. In fact, the strong equivalence principle is what makes general relativity "General Relativity". The reason is that the gravitational component of the theory is actually a geometric part (mainly established by the presumed validity of the weak equivalence principle and it represents a generalization of the classical gravity) that sets the scene for the physical part and the applicability of this physical part (which is the important part for the "General Theory")

is the essence of the strong equivalence principle.
Note: the form of the physical laws in any frame can be determined by transforming the laws from the freely falling frames (in which special relativity applies) to the other frames, and hence non-inertial effects (caused by gravity or/and acceleration) should be accounted for by these transformations. This requires the availability of general transformations and is facilitated by casting the special relativistic laws in tensor forms (see exercise 3).

2. Give examples of the features of general relativity that characterize its nature as a "General Theory".
Answer: For example:
• The validity of the strong equivalence principle.
• The employment of general coordinate transformations instead of the Lorentz transformations.
• The use of tensor derivatives (i.e. covariant and absolute) instead of ordinary derivatives (i.e. partial and total) in the formulation of the laws of physics.[113]
• The imposition of the locality condition in the application of the laws of physics which leads to the use of local inertial frames (or freely falling frames).
3. How to generalize a special relativistic law to become a general relativistic law?
Answer: The procedure of this generalization is outlined in the following points:
• Because the law is supposed to be a special relativistic law, we use the known formulation of the law in a Minkowskian spacetime coordinated by an inertial frame.
• The special relativistic formulation of the law should then be cast in an invariant tensorial form (e.g. by converting partial and total derivatives to covariant and absolute derivatives) if it is not already so.
• By the general invariance (or general covariance) principle of general relativity and thanks to the tensorial nature of the new form, the law in its new tensorial form should be valid in any other frame (inertial or accelerating or gravitating) coordinating any type of spacetime (whether flat or curved).[114]
Therefore, all we need to do to find out how the law should look like in any other frame is to find a valid general transformation from the current frame (which is inertial coordinating flat spacetime)[115] to the other frame (which could be of any type coordinating any type of spacetime).
Note: the above procedure essentially describes the general relativistic recipe for introducing the effect of gravity onto a physical system. In fact, the essence of this procedure is supposed to be contained in the application of the *Principle of General Covariance* as described in the Problems of § 1.8.2. Also see exercise 2 of 10.1.11.
4. Analyze and assess the procedure proposed in the previous exercise.
Answer: We note the following:
• The above procedure reduces the problem of finding the physical laws in different types of frames (coordinating various types of spacetime) to a rather simple problem of finding the appropriate coordinate transformations between these frames.
• The tensorial nature of the formulation of the law may justify the transformation (invariantly) from one frame to another frame where the two frames are coordinating the same spacetime (or at least the same type of spacetime). But the tensorial nature may not be sufficient to justify this transformation (assuming such a transformation does exist) when the nature of the spacetime of the two frames is different (i.e. one is flat and the other is curved) unless the difference between the nature of the frame and the nature of the spacetime is abolished by incorporating the nature of the spacetime in the frame itself and hence the curvature and flatness of spacetime becomes an aspect of the frame (which may be justified by the equivalence principle). Yes, the locality condition (which ensures local flatness) should address this issue but this should put limit on the general validity and applicability.
• The mathematical and theoretical validity of this procedure does not imply its physical validity because the latter depends on the experimental evidence by verifying that the general relativistic laws obtained by this procedure are valid physical laws in the presumed physical situations. As we outlined earlier, the invariance of laws has physical content side and formal side and the proposed procedure

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[113] In fact, the use of tensor derivatives should be seen as an instance of the use of tensor forms and rules in general.

[114] In fact, this should require the principle of equivalence as well (as may be implied by "general covariance").

[115] In this context, "current frame" (in which special relativity supposedly applies) should refer to the freely falling local frame which is effectively inertial coordinating a flat patch of spacetime.

seems to address the latter (as suggested by the use of tensor formulation) but not the former, i.e. even if we find the appropriate transformation (that transforms invariantly in this formal sense) we may not necessarily find the appropriate physical law in the other frame.

• In fact, even the theoretical validity (as well as sensibility and viability) of this procedure requires justification because it is not clear that the underlying general relativistic rationale is logically consistent and valid and the procedure is viable, implementable and general (regardless of its physical validity and value).

• In brief, the availability of general transformations does not logically imply the actual transformation of the physical effect from one frame to the other (because the frames may differ in such a way that affects the transformed law in a more fundamental way than that represented by the coordinate transformations); moreover the existence of such simple transformations may not be guaranteed (even theoretically and mathematically let alone practically).

Chapter 4
Solutions of the Field Equation

1. We may describe a solution of a scientific problem as "ideal solution". Discuss this issue and indicate the approximation methods that are generally used in science to obtain second best solutions.
   Answer: An ideal solution of an analytically formulated scientific problem is an exact analytical solution that is based on observing all the stated conditions and assumptions of the problem rigorously. Such a solution is usually obtainable in very simple and trivial scientific problems, so in most cases obtaining an ideal solution is either very difficult or impossible. Accordingly, science in general is based on finding and accepting second best solutions and that is what makes the continuity and progress of science possible. There are many strategies and methods that are followed systematically by scientists across all branches of science to obtain approximate solutions, and in fact many of these have deep roots in the intellectual evolution of mankind since ancient times. Some of these approximation methods are:
   • Relaxing the conditions and assumptions on which the actual problem is based and hence in essence we do not solve the problem itself but we solve a similar problem that is easier to solve because it is based on less restricted conditions and more simple assumptions. For example, instead of solving the problem of the gravitational field of the actual Earth with all its complex shape, structure and composition we solve it as a problem of a perfect sphere of spherically symmetric density or uniform density or even as a massive particle.
   • Using approximate analytical methods such as ignoring some terms because their effect is minor. A prominent example of this approach is the use of power series approximation where an analytical expression in the problem is reduced to a power series which is then truncated by discarding the high order terms whose contribution is negligible according to the required accuracy. We may also include in this category using linearization and perturbation methods.
   • Using numerical methods and approximations. This approach thrived in a spectacular way during the last decades and hence it invaded all scientific disciplines thanks to the invention of computers and the huge advancement in computational methods and techniques which facilitate the implementation and application of highly reliable numerical methods like finite element and finite difference.
   Note: the present book contains many examples of the use of the above approximation methods.
2. Outline some of the approximation methods that are used to overcome the technical difficulties in solving general relativistic problems.
   Answer: The methods given in the answer of the previous exercise are examples of the general approximation methods that are used in solving general relativistic problems as in any other scientific subject.[116] More specialized methods that depend on the nature of the problems are also used, e.g. using classical and special relativistic guiding limits or using approximations based on locality or equivalence. However, some of these specialized methods are built-in within the theoretical structure of the theory itself and hence they are not approximation methods in a strict technical sense although they in essence are approximations since they play the role of providing simplified prototypes for the real situations and settings which are more sophisticated.
3. What "solving the Field Equations" means?
   Answer: The Field Equations are a set of differential equations where the unknown in these equations is the metric tensor of the spacetime while the known is the energy-momentum tensor.[117] So, solving the Field Equations means finding the unknown metric tensor which is embedded in the Einstein tensor since the Einstein tensor is synthesized from the metric tensor (see § 2.13). Now, because the metric tensor uniquely and completely determines the geometry of the spacetime then all we need to know

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[116] In this context we should mention an important technique that we discussed in detail earlier which is the use of the linearized general relativity (see § 3.4).

[117] As indicated earlier, this represents a rather ideal situation.

about this geometry (which describes gravity according to general relativity) can be obtained from the metric tensor. As indicated earlier, one of the ultimate goals of solving the Field Equations is to find the geodesic trajectories and these geodesic trajectories are obtained from the metric tensor since the geodesic equation (see § 2.9.5) solely depends on the Christoffel symbols (see § 2.6) which solely depend on the metric tensor. In fact, practically all the consequences of general relativistic gravity are derived directly or indirectly from the metric of the spacetime (see for example § 8) although some can also be obtained from certain fundamental principles (such as the equivalence principle).

4. Outline the steps that should be followed in formulating and solving gravity problems according to general relativity.
   Answer: The steps are:[118]
   - Obtain the energy-momentum tensor that describes the physical setting of the problem.
   - Form the Field Equations using the obtained energy-momentum tensor.
   - Solve the Field Equations to obtain the metric tensor of the spacetime.
   - Obtain from the metric tensor what is required in the problem such as finding the geodesic and null geodesic trajectories.
5. Outline some alternative tackling strategies that are used in formulating and solving general relativistic problems.
   Answer: The ideal strategy in formulating and solving general relativistic problems is outlined in the answer of the previous exercise where we start from formulating the energy-momentum tensor and end with finding the metric tensor and subsequent consequences like geodesic trajectories. However, in most cases this strategy is not viable and hence alternative strategies should be followed. For example:
   - We may start with an initial guess (guided for instance by classical physics or simplified assumptions or symmetries and analogies) about the relevant metric tensor that corresponds to the physical setting of the problem. The energy-momentum tensor that corresponds to the guessed metric tensor may then be generated with the help of the guessed metric.[119] The generated energy-momentum tensor can then be elaborated and used in a similar manner to that of the ideal strategy. This may be followed by checking if the solution obtained by this guess is correct (as known for instance from classical theory) or acceptable (according to some indications like converging to certain known limits such as special relativistic limits). The guess or/and solution can then be improved iteratively if such improvement is needed and viable until we reach a satisfactory solution within the existing theoretical and practical requirements and limitations.

     We note that in our derivation of the Schwarzschild metric (see § 4.1.1) we followed a similar strategy where we started from an informed initial guess about the metric tensor (using our knowledge of the simple physical setting and its symmetries to make this guess). However, due to the relative simplicity of the Schwarzschild problem and because it is a vacuum problem we did not need to go through the generation and refinement of the energy-momentum tensor. Instead, we used some simple arguments and reasoning (with the immediate employment of the vacuum Field Equation) to find the final and definite form of the metric tensor that corresponds to the initial guess.
   - We may also start with a guess about both the metric tensor and the energy-momentum tensor and follow a similar procedure to the procedure of the previous strategy. Examples of this strategy (or similar to this strategy) could be found in solving some cosmological problems to generate mathematical models for the Universe and its evolution.

   Note 1: there is a major limitation about most of the above-described methods and strategies of solving general relativistic problems that is these methods apply to simple physical problems where the gravitated objects are test particles. As we will see later (refer for example to § 10.1.14), in more complex situations where the gravitated objects are not test particles there is a circularity problem and hence more complex and less general methods and strategies (e.g. using iterative methods) may be applied. However, their generality and validity are not guaranteed (and hence some problems may be impossible to solve general-relativistically by any known method).

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[118] Again, this ideal approach is hardly followed in real life.

[119] In vacuum problems, the energy-momentum tensor vanishes and hence the procedure is simplified. So, all we need to consider is the metric and its companions.

Note 2: we may also add another tackling strategy which is the use of the linearized form of the Field Equation (see § 3.4). However, this tackling strategy is of different nature to the above strategies and it should be considered more appropriately as an approximation method rather than a tackling strategy. This also applies to other similar "tackling strategies" which are essentially approximation methods.

4.1 Schwarzschild Solution

4.1.1 Derivation of Schwarzschild Metric

1. What is the Schwarzschild solution?
   Answer: It is the metric of the spacetime outside a static, spherically symmetric and electrically uncharged massive object.
2. Summarize the main characteristics of the Schwarzschild solution.
   Answer: We note the following:
   - Historically, it is the first solution to the Field Equation.
   - It is one of the simplest (and possibly the simplest) non-trivial analytical solutions to the Field Equation.
   - It is one of the most important (and possibly the most important) solutions to the Field Equation.
   - It is spherically symmetric and time independent.
   - It describes the geometry of spacetime outside a static and electrically neutral gravitating object with spherically symmetric mass density.
   - It is a vacuum solution, i.e. a solution of $R_{\mu\nu} = 0$.
   - It converges to the Lorentz metric of special relativity at infinity, i.e. $r \to \infty$.
   - It converges to the Lorentz metric of special relativity when mass vanishes, i.e. $M = 0$.
   - It has a coordinate singularity at the Schwarzschild radius.
   - It provides the basis for many consequences and predictions of general relativity such as gravitational time dilation (see § 8.3) and black holes (see § 8.7).
   - Any solution of the Schwarzschild problem is equivalent to the Schwarzschild solution (i.e. the solution is unique).
   - It is a solution even to some problems in which the source of gravity is not static (according to the Birkhoff theorem) or not spherically symmetric.
   - The gravitating object in the Schwarzschild solution is characterized by mass M only.[120]

   Note: some of the above characteristics belong to the Schwarzschild solution in the Schwarzschild coordinates specifically.
3. Summarize the procedure that we followed to find the Schwarzschild metric.
   Answer: The procedure is outlined in the following points:
   - We started from an initial guess (based on certain assumptions, simplifications, symmetries and boundary conditions) that we obtained from analyzing the physical situation of the problem. According to the guess, the metric tensor is a time independent and spherically symmetric diagonal tensor with its g_{00} and g_{11} components being dependent on two functions $H(r)$ and $K(r)$ while its g_{22} and g_{33} components being the same as the θ and ϕ components of the metric tensor of a spatial spherical coordinate system (but with negative sign), that is:

   $$[g_{\mu\nu}] = \text{diag}\left[e^{H},\ -e^{K},\ -r^{2},\ -r^{2}\sin^{2}\theta\right]$$

   - From the guessed metric tensor we obtained the non-vanishing Christoffel symbols using the equation $\Gamma^{\xi}_{\mu\nu} = \frac{g^{\xi\alpha}}{2}\left(\partial_\nu g_{\mu\alpha} + \partial_\mu g_{\nu\alpha} - \partial_\alpha g_{\mu\nu}\right)$. However, the obtained Christoffel symbols are still not determined completely since they contain the yet-unknown functions $H(r)$ and $K(r)$.
   - From the (semi-determined) Christoffel symbols we obtained the non-vanishing components of the Ricci curvature tensor $R_{\mu\nu}$ using the equation $R_{\mu\nu} = \partial_\nu \Gamma^{\alpha}_{\mu\alpha} - \partial_\alpha \Gamma^{\alpha}_{\mu\nu} + \Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\beta\nu} - \Gamma^{\beta}_{\mu\nu}\Gamma^{\alpha}_{\beta\alpha}$.

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[120] In fact, this is from a physical perspective. From a geometric perspective the object is also characterized by other attributes like spherical symmetry.

- Based on the vacuum Field Equation, we formed the system of equations $R_{\mu\nu} = 0$.
- We solved the system of equations $R_{\mu\nu} = 0$ simultaneously and hence we obtained a definite form (up to a yet-undetermined constant C_s) for the functions $H(r)$ and $K(r)$.
- We used the Newtonian gravity theory in the classical limit to determine C_s and hence obtain the final and completely definite form of the Schwarzschild metric which is:

$$[g_{\mu\nu}] = \text{diag}\left[1 - \frac{2GM}{c^2 r},\ -\left(1 - \frac{2GM}{c^2 r}\right)^{-1},\ -r^2,\ -r^2 \sin^2\theta\right]$$

4. Write down the Schwarzschild metric tensor and the Schwarzschild quadratic form explaining all the symbols used.
Answer:

$$\begin{aligned}[][g_{\mu\nu}] &= \text{diag}\left[1 - \frac{2GM}{c^2 r},\ -\left(1 - \frac{2GM}{c^2 r}\right)^{-1},\ -r^2,\ -r^2 \sin^2\theta\right] \\ (d\sigma)^2 &= \left(1 - \frac{2GM}{c^2 r}\right)(cdt)^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1}(dr)^2 - r^2(d\theta)^2 - r^2\sin^2\theta\,(d\phi)^2\end{aligned}$$

where $[g_{\mu\nu}]$ is the metric tensor, $(d\sigma)^2$ is the quadratic form of the line element $d\sigma$ of the 4D spacetime, (ct, r, θ, ϕ) are the Schwarzschild coordinates, c is the speed of light, G is the gravitational constant, M is the mass of the Schwarzschild object, and diag means diagonal matrix (which represents the tensor). We note that the interpretation of the Schwarzschild coordinates and their relation to the physical coordinates will be discussed later (see § 6.2, § 6.3.2 and § 6.4.2).

5. Write the Schwarzschild metric tensor in terms of the escape speed v_{es} and in terms of the Schwarzschild radius R_S.
Answer: We have:

$$v_{es} = \sqrt{\frac{2GM}{r}} \qquad \text{and} \qquad R_S = \frac{2GM}{c^2}$$

Hence:

$$\begin{aligned}[][g_{\mu\nu}] &= \text{diag}\left[1 - \frac{v_{es}^2}{c^2},\ -\left(1 - \frac{v_{es}^2}{c^2}\right)^{-1},\ -r^2,\ -r^2 \sin^2\theta\right] \\ [g_{\mu\nu}] &= \text{diag}\left[1 - \frac{R_S}{r},\ -\left(1 - \frac{R_S}{r}\right)^{-1},\ -r^2,\ -r^2 \sin^2\theta\right]\end{aligned}$$

6. Verify that the Schwarzschild metric converges to the Lorentz metric of special relativity very far away from the gravitating object.
Answer: The Schwarzschild metric tensor is given by:

$$[g_{\mu\nu}] = \text{diag}\left[1 - \frac{2GM}{c^2 r},\ -\left(1 - \frac{2GM}{c^2 r}\right)^{-1},\ -r^2,\ -r^2 \sin^2\theta\right]$$

At very far away from the gravitating object the radial coordinate r is very large (i.e. $r \to \infty$) and hence the term $\frac{2GM}{c^2 r}$ in the metric coefficients tends to zero. Thus, the Schwarzschild metric tensor becomes:

$$[g_{\mu\nu}] = \text{diag}\left[1,\ -1,\ -r^2,\ -r^2 \sin^2\theta\right]$$

which is the Lorentz metric of special relativity (using spatial spherical coordinate system; see exercise 10 of § 2.5).

7. Verify that when the mass vanishes the Schwarzschild metric converges to the Lorentz metric of special relativity.
   Answer: The Schwarzschild metric tensor is given by:
   $$[g_{\mu\nu}] = \operatorname{diag}\left[1 - \frac{2GM}{c^2 r},\ -\left(1 - \frac{2GM}{c^2 r}\right)^{-1},\ -r^2,\ -r^2 \sin^2\theta\right]$$
   When the mass vanishes, $M = 0$ and hence:
   $$[g_{\mu\nu}] = \operatorname{diag}\left[1,\ -1,\ -r^2,\ -r^2 \sin^2\theta\right]$$
   which is the Lorentz metric of special relativity (using spatial spherical coordinate system; see exercise 10 of § 2.5).
8. Based on the last two exercises, analyze the relation between special relativity and general relativity and how the latter is seen to converge to the former in the absence of gravity.
   Answer: From the Field Equations we get the metric of spacetime, and this metric converges to the metric of flat spacetime in the absence of gravity (either at infinity as in exercise 6 or by the absence of matter as in exercise 7). This flat spacetime is the spacetime of special relativity and in this sense general relativity converges to special relativity in the absence of gravity (i.e. the curved spacetime of general relativity reduces to the flat spacetime of special relativity). However, this does not mean that the formalism of general relativity (as represented by the Field Equations and their consequences) converges to the formalism of special relativity (as represented by Lorentz transformations and their consequences) in the absence of gravity. Accordingly, the application of special relativity in the absence of gravity is based on the theory of special relativity and not on the theory of general relativity as a gravity theory. In other words, it is a condition imposed from outside the theory of general relativity (as a gravity theory) and hence in this sense it is arbitrary and is not based on a natural convergence of the formalism of general relativity (as it is the case for example in the convergence of special relativity to classical mechanics at low speed; see B4). So, we can say that in general relativity we actually have two independent formalisms: the formalism of gravity (represented by the Field Equations and their consequences) and the formalism of special relativity (represented by the rules of Lorentz mechanics which are ultimately based on the Lorentz transformations).
   We should also note that the convergence of general relativity to special relativity (even in the above restricted sense) is based on the premise that the physical effects are results of (or depend on) the curvature and flatness of spacetime.[121]
9. What are the types of singularity and how you define them?
   Answer: We have two main types of singularity: accidental (or coordinate) singularity and essential (or inherent) singularity. The first is casual and originates from the use of a particular coordinate system and hence it is an unfortunate mathematical artifact that can be removed by changing the coordinate system, while the second originates from an inherent mathematical situation and hence it cannot be removed by changing the coordinate system.
   We should remark that being a coordinate singularity does not necessarily mean it has no physical significance because coordinate singularity can still characterize the solution and have physical meaning, i.e. a solution based on a particular coordinate system can demonstrate a physical feature of the problem that is not demonstrated by a solution based on another coordinate system.
   We should also remark that the above classification of singularity is mainly based on formal or mathematical considerations. Accordingly, we may add (in our opinion and terminology) a third type of singularity which we call physical singularity, i.e. singularity in the real world regardless of any mathematical requirement. As we indicated previously, mathematical singularity does not imply physical singularity (i.e. even essential mathematical singularity could be a mere artifact that does not reflect

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[121] In fact, this premise (when applied to the local application of special relativity in the curved spacetime) requires justification especially if the equivalence principle (at least in its strong form) is not validated. However, this is a rather different issue from the issue of this exercise.

the physical reality). Moreover, we do not believe in the existence of physical singularity because all the entities in this world are limited and finite (according to our experiences and observations).

10. Show that there is a singularity in the Schwarzschild quadratic form. What is the physical interpretation of this? What is the type of this singularity?

Answer: The Schwarzschild quadratic form is given by:

$$(d\sigma)^2 = \left(1 - \frac{R_S}{r}\right)(cdt)^2 - \left(1 - \frac{R_S}{r}\right)^{-1}(dr)^2 - r^2 (d\theta)^2 - r^2 \sin^2\theta\,(d\phi)^2 \tag{17}$$

where R_S ($\equiv \frac{2GM}{c^2}$) is the Schwarzschild radius. So, if $r = R_S$ then we have a singularity since the second term includes division by zero. In fact, this is based on the assumption that the domain of the Schwarzschild solution includes $r = R_S$; otherwise there is no singularity at $r = R_S$ since it is not in the domain of the solution. The inclusion of $r = R_S$ depends on the nature of the gravitating object as being black hole or not. For ordinary objects (such as stars) the domain does not include $r = R_S$ because the actual radius of the object is much larger than the Schwarzschild radius R_S and hence the Schwarzschild solution (as a vacuum solution that applies only to the region outside the actual radius of the object) does not apply to $r = R_S$. So, this singularity applies to black holes only.

A potential physical interpretation of this singularity is that when the radius of the gravitating object is reduced to the Schwarzschild radius then we have a black hole (see § 8.7).

This singularity is an accidental (or coordinate) singularity because it can be removed by using a different coordinate system, e.g. the Kruskal-Szekeres coordinates as shown in the literature. As indicated in the previous exercise, being a coordinate singularity does not affect its physical interpretation (as signaling the shift to a black hole) because coordinate singularity can have physical significance (i.e. it could reflect a certain physical feature of the system even though it is not a "real" singularity in a certain sense).

Note: this question and its answer ignore the singularity at $r = 0$ because it is not within the domain of the Schwarzschild solution which corresponds to the vacuum region outside the gravitating object and hence it does not include $r = 0$ where the gravitating object is located.

11. Show that the non-vanishing Christoffel symbols of the Schwarzschild spacetime (in the Schwarzschild coordinates) are those given by Eq. 126 (in the text) and verify that these symbols are as given by Eq. 126.

Answer: First, we note that in the above Schwarzschild solution (i.e. Schwarzschild metric based on the Schwarzschild coordinates) we are using orthogonal coordinates since the metric tensor is diagonal (see exercise 15 of § 2.5) and hence the formulae of the Christoffel symbols are simplified as shown in Problem 6 of § 2.6. Also, the coefficients of the metric tensor are given by:

$$\begin{array}{llll} g_{00} = e^H & g_{11} = -e^K & g_{22} = -r^2 & g_{33} = -r^2 \sin^2\theta \\ g_{\mu\nu} = 0 & (\mu \neq \nu) & & \end{array}$$

Now, in the 4D spacetime we have $n^3 = 64$ Christoffel symbols (see § 2.6). Considering the identicality and difference of the indices of the Christoffel symbols, we have 4 main cases:

(**a**) All the indices are different: this represents $n(n-1)(n-2) = 24$ symbols which is the number of non-repetitive permutations. Due to the orthogonality of the system, all these symbols are zero. Hence, the following 24 symbols are zero:

$$\begin{array}{llllll} \Gamma^0_{12} & \Gamma^0_{21} & \Gamma^0_{13} & \Gamma^0_{31} & \Gamma^0_{23} & \Gamma^0_{32} \\ \Gamma^1_{02} & \Gamma^1_{20} & \Gamma^1_{03} & \Gamma^1_{30} & \Gamma^1_{23} & \Gamma^1_{32} \\ \Gamma^2_{01} & \Gamma^2_{10} & \Gamma^2_{03} & \Gamma^2_{30} & \Gamma^2_{13} & \Gamma^2_{31} \\ \Gamma^3_{01} & \Gamma^3_{10} & \Gamma^3_{02} & \Gamma^3_{20} & \Gamma^3_{12} & \Gamma^3_{21} \end{array}$$

(**b**) Only the two paired indices are identical: this represents $n(n-1) = 12$ symbols, i.e. n non-paired times $(n-1)$ paired (or the other way around). Due to the orthogonality of the system, these symbols are given by:

$$\Gamma^{\nu}_{\mu\mu} = -\frac{1}{2g_{\nu\nu}}\partial_{\nu} g_{\mu\mu} \qquad (\mu \neq \nu,\ \text{no sum on } \mu \text{ or } \nu)$$

Now, $\partial_{\nu} g_{\mu\mu} = 0$ (and hence $\Gamma^{\nu}_{\mu\mu} = 0$) if $g_{\mu\mu}$ is independent of the ν^{th} coordinate. Hence, the following 8 symbols are zero:

$$\begin{array}{lll}
\Gamma^{0}_{11} & \Gamma^{0}_{22} & \Gamma^{0}_{33} \\
\bigcirc & \bigcirc & \bigcirc \\
\Gamma^{2}_{00} & \Gamma^{2}_{11} & \bigcirc \\
\Gamma^{3}_{00} & \Gamma^{3}_{11} & \Gamma^{3}_{22}
\end{array}$$

Regarding the remaining 4 symbols (marked with $\bigcirc$), we have:

$$\begin{aligned}
\Gamma^{1}_{00} &= -\frac{1}{2g_{11}}\partial_1 g_{00} = -\frac{1}{2\left(-e^{K}\right)}\partial_r e^{H} = \frac{e^{H}}{2e^{K}}\frac{dH}{dr} = \frac{1}{2}e^{H-K}H' \\
\Gamma^{1}_{22} &= -\frac{1}{2g_{11}}\partial_1 g_{22} = -\frac{1}{2\left(-e^{K}\right)}\partial_r\left(-r^2\right) = \frac{2r}{-2e^{K}} = -re^{-K} \\
\Gamma^{1}_{33} &= -\frac{1}{2g_{11}}\partial_1 g_{33} = -\frac{1}{2\left(-e^{K}\right)}\partial_r\left(-r^2\sin^2\theta\right) = \frac{2r\sin^2\theta}{-2e^{K}} = -re^{-K}\sin^2\theta \\
\Gamma^{2}_{33} &= -\frac{1}{2g_{22}}\partial_2 g_{33} = -\frac{1}{2\left(-r^2\right)}\partial_\theta\left(-r^2\sin^2\theta\right) = \frac{2r^2\sin\theta\cos\theta}{-2r^2} = -\sin\theta\cos\theta
\end{aligned}$$

(**c**) Only two non-paired indices are identical: this represents $2n(n-1) = 24$ symbols, i.e. n identical times $(n-1)$ different (or the other way around) times 2 possibilities for the order of the paired indices. Due to the orthogonality of the system, these symbols are given by:

$$\Gamma^{\mu}_{\mu\nu} = \Gamma^{\mu}_{\nu\mu} = \frac{1}{2g_{\mu\mu}}\partial_{\nu} g_{\mu\mu} \qquad (\mu \neq \nu,\ \text{no sum on } \mu)$$

Now, $\partial_{\nu} g_{\mu\mu} = 0$ (and hence $\Gamma^{\mu}_{\mu\nu} = 0$) if $g_{\mu\mu}$ is independent of the ν^{th} coordinate. Hence, the following 16 symbols are zero:

$$\begin{array}{ccc}
\bigcirc & \Gamma^{0}_{02} = \Gamma^{0}_{20} & \Gamma^{0}_{03} = \Gamma^{0}_{30} \\
\Gamma^{1}_{10} = \Gamma^{1}_{01} & \Gamma^{1}_{12} = \Gamma^{1}_{21} & \Gamma^{1}_{13} = \Gamma^{1}_{31} \\
\Gamma^{2}_{20} = \Gamma^{2}_{02} & \bigcirc & \Gamma^{2}_{23} = \Gamma^{2}_{32} \\
\Gamma^{3}_{30} = \Gamma^{3}_{03} & \bigcirc & \bigcirc
\end{array}$$

Regarding the remaining 8 symbols (marked with $\bigcirc$), we have:

$$\begin{aligned}
\Gamma^{0}_{01} = \Gamma^{0}_{10} &= \frac{1}{2g_{00}}\partial_1 g_{00} = \frac{1}{2e^{H}}\partial_r e^{H} = \frac{e^{H}}{2e^{H}}\frac{dH}{dr} = \frac{1}{2}H' \\
\Gamma^{2}_{21} = \Gamma^{2}_{12} &= \frac{1}{2g_{22}}\partial_1 g_{22} = \frac{1}{2\left(-r^2\right)}\partial_r\left(-r^2\right) = \frac{-2r}{-2r^2} = \frac{1}{r} \\
\Gamma^{3}_{31} = \Gamma^{3}_{13} &= \frac{1}{2g_{33}}\partial_1 g_{33} = \frac{1}{2\left(-r^2\sin^2\theta\right)}\partial_r\left(-r^2\sin^2\theta\right) = \frac{-2r\sin^2\theta}{-2r^2\sin^2\theta} = \frac{1}{r} \\
\Gamma^{3}_{32} = \Gamma^{3}_{23} &= \frac{1}{2g_{33}}\partial_2 g_{33} = \frac{1}{2\left(-r^2\sin^2\theta\right)}\partial_\theta\left(-r^2\sin^2\theta\right) = \frac{-2r^2\sin\theta\cos\theta}{-2r^2\sin^2\theta} = \cot\theta
\end{aligned}$$

(**d**) All the indices are identical: this represents $n = 4$ symbols since we have 4 possible values for the identical index. Due to the orthogonality of the system, these symbols are given by:

$$\Gamma^{\mu}_{\mu\mu} = \frac{1}{2g_{\mu\mu}}\partial_{\mu} g_{\mu\mu} \qquad (\text{no sum on } \mu)$$

Now, $\partial_\mu g_{\mu\mu} = 0$ (and hence $\Gamma^\mu_{\mu\mu} = 0$) if $g_{\mu\mu}$ is independent of the μ^{th} coordinate. Hence, the following 3 symbols are zero:

$$\Gamma^0_{00} \qquad \bigcirc \qquad \Gamma^2_{22} \qquad \Gamma^3_{33}$$

Regarding the remaining symbol (marked with ◯), we have:

$$\Gamma^1_{11} = \frac{1}{2g_{11}}\partial_1 g_{11} = \frac{1}{2\left(-e^K\right)}\partial_r\left(-e^K\right) = \frac{-e^K}{-2e^K}\frac{dK}{dr} = \frac{1}{2}K'$$

Note: partial derivative becomes total derivative when the differentiated function has a single dependency (i.e. it is a function of only the variable of differentiation). For example, $\frac{\partial H}{\partial r} = \frac{\partial H}{\partial r}\frac{dr}{dr} = \frac{dH}{dr}$ because H is a function of r only.

12. Show that the non-vanishing coefficients of the Ricci curvature tensor of the Schwarzschild spacetime (in the Schwarzschild coordinates) are R_{00}, R_{11}, R_{22} and R_{33} and verify that these coefficients are as given by Eqs. 127-130 (in the text).

Answer:[122] We use the equation: $R_{\mu\nu} = \partial_\nu \Gamma^\alpha_{\mu\alpha} - \partial_\alpha \Gamma^\alpha_{\mu\nu} + \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha} - \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu}$. We also use other results that we obtained before (e.g. the partial derivative with respect to time ∂_0 is zero because the metric is time independent, or some Christoffel symbols like Γ^2_{00} are identically zero, or some Christoffel symbols are independent of some coordinates like the independence of Γ^3_{13} of θ). However, for the sake of brevity we do not provide detailed justifications (as we usually do).

- $R_{00} = \partial_0 \Gamma^\alpha_{0\alpha} - \partial_\alpha \Gamma^\alpha_{00} + \Gamma^\alpha_{\beta 0}\Gamma^\beta_{0\alpha} - \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{00}$

$$\begin{aligned}
\partial_0 \Gamma^\alpha_{0\alpha} &= \partial_0 \Gamma^0_{00} + \partial_0 \Gamma^1_{01} + \partial_0 \Gamma^2_{02} + \partial_0 \Gamma^3_{03} \\
&= 0 + 0 + 0 + 0 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\partial_\alpha \Gamma^\alpha_{00} &= \partial_0 \Gamma^0_{00} + \partial_1 \Gamma^1_{00} + \partial_2 \Gamma^2_{00} + \partial_3 \Gamma^3_{00} \\
&= 0 + \partial_r\left(\frac{1}{2}e^{H-K}H'\right) + 0 + 0 \\
&= \frac{1}{2}e^{H-K}\left(H' - K'\right)H' + \frac{1}{2}e^{H-K}H'' \\
&= \frac{1}{2}e^{H-K}\left[\left(H'\right)^2 - H'K' + H''\right]
\end{aligned}$$

$$\begin{aligned}
\Gamma^\alpha_{\beta 0}\Gamma^\beta_{0\alpha} &= \Gamma^0_{\beta 0}\Gamma^\beta_{00} + \Gamma^1_{\beta 0}\Gamma^\beta_{01} + \Gamma^2_{\beta 0}\Gamma^\beta_{02} + \Gamma^3_{\beta 0}\Gamma^\beta_{03} \\
&= \Gamma^0_{00}\Gamma^0_{00} + \Gamma^1_{00}\Gamma^0_{01} + \Gamma^2_{00}\Gamma^0_{02} + \Gamma^3_{00}\Gamma^0_{03} + \\
&\quad \Gamma^0_{10}\Gamma^1_{00} + \Gamma^1_{10}\Gamma^1_{01} + \Gamma^2_{10}\Gamma^1_{02} + \Gamma^3_{10}\Gamma^1_{03} + \\
&\quad \Gamma^0_{20}\Gamma^2_{00} + \Gamma^1_{20}\Gamma^2_{01} + \Gamma^2_{20}\Gamma^2_{02} + \Gamma^3_{20}\Gamma^2_{03} + \\
&\quad \Gamma^0_{30}\Gamma^3_{00} + \Gamma^1_{30}\Gamma^3_{01} + \Gamma^2_{30}\Gamma^3_{02} + \Gamma^3_{30}\Gamma^3_{03} \\
&= 0 + \frac{1}{2}e^{H-K}H'\frac{1}{2}H' + 0 + 0 + \\
&\quad \frac{1}{2}H'\frac{1}{2}e^{H-K}H' + 0 + 0 + 0 + \\
&\quad 0 + 0 + 0 + 0 + \\
&\quad 0 + 0 + 0 + 0 \\
&= \frac{1}{2}e^{H-K}\left(H'\right)^2
\end{aligned}$$

---

[122] The answer of this question is exceptionally long. In fact, there are more compact methods for obtaining the required results. However, we prefer this long method for simplicity, clarity and to provide practice on basic tensor operations. This should also apply (to some extent) to the answer of the previous question.

$$\begin{aligned}
\Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{00} &= \Gamma^{0}_{\beta 0}\Gamma^{\beta}_{00} + \Gamma^{1}_{\beta 1}\Gamma^{\beta}_{00} + \Gamma^{2}_{\beta 2}\Gamma^{\beta}_{00} + \Gamma^{3}_{\beta 3}\Gamma^{\beta}_{00} \\
&= \Gamma^{0}_{00}\Gamma^{0}_{00} + \Gamma^{1}_{01}\Gamma^{0}_{00} + \Gamma^{2}_{02}\Gamma^{0}_{00} + \Gamma^{3}_{03}\Gamma^{0}_{00} + \\
&\quad \Gamma^{0}_{10}\Gamma^{1}_{00} + \Gamma^{1}_{11}\Gamma^{1}_{00} + \Gamma^{2}_{12}\Gamma^{1}_{00} + \Gamma^{3}_{13}\Gamma^{1}_{00} + \\
&\quad \Gamma^{0}_{20}\Gamma^{2}_{00} + \Gamma^{1}_{21}\Gamma^{2}_{00} + \Gamma^{2}_{22}\Gamma^{2}_{00} + \Gamma^{3}_{23}\Gamma^{2}_{00} + \\
&\quad \Gamma^{0}_{30}\Gamma^{3}_{00} + \Gamma^{1}_{31}\Gamma^{3}_{00} + \Gamma^{2}_{32}\Gamma^{3}_{00} + \Gamma^{3}_{33}\Gamma^{3}_{00} \\
&= 0 + 0 + 0 + 0 + \\
&\quad \frac{1}{2}H'\frac{1}{2}e^{H-K}H' + \frac{1}{2}K'\frac{1}{2}e^{H-K}H' + \frac{1}{r}\frac{1}{2}e^{H-K}H' + \frac{1}{r}\frac{1}{2}e^{H-K}H' + \\
&\quad 0 + 0 + 0 + 0 + \\
&\quad 0 + 0 + 0 + 0 \\
&= e^{H-K}\left[\frac{1}{4}\left(H'\right)^{2} + \frac{1}{4}H'K' + \frac{1}{r}H'\right]
\end{aligned}$$

$$\begin{aligned}
R_{00} &= \partial_{0}\Gamma^{\alpha}_{0\alpha} - \partial_{\alpha}\Gamma^{\alpha}_{00} + \Gamma^{\alpha}_{\beta 0}\Gamma^{\beta}_{0\alpha} - \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{00} \\
&= 0 - \frac{1}{2}e^{H-K}\left[\left(H'\right)^{2} - H'K' + H''\right] + \frac{1}{2}e^{H-K}\left(H'\right)^{2} - \\
&\quad e^{H-K}\left[\frac{1}{4}\left(H'\right)^{2} + \frac{1}{4}H'K' + \frac{1}{r}H'\right] \\
&= e^{H-K}\left(\frac{H'K'}{4} - \frac{\left(H'\right)^{2}}{4} - \frac{H''}{2} - \frac{H'}{r}\right)
\end{aligned}$$

- $R_{11} = \partial_{1}\Gamma^{\alpha}_{1\alpha} - \partial_{\alpha}\Gamma^{\alpha}_{11} + \Gamma^{\alpha}_{\beta 1}\Gamma^{\beta}_{1\alpha} - \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{11}$

$$\begin{aligned}
\partial_{1}\Gamma^{\alpha}_{1\alpha} &= \partial_{1}\Gamma^{0}_{10} + \partial_{1}\Gamma^{1}_{11} + \partial_{1}\Gamma^{2}_{12} + \partial_{1}\Gamma^{3}_{13} \\
&= \partial_{r}\left(\frac{1}{2}H'\right) + \partial_{r}\left(\frac{1}{2}K'\right) + \partial_{r}\left(\frac{1}{r}\right) + \partial_{r}\left(\frac{1}{r}\right) \\
&= \frac{1}{2}H'' + \frac{1}{2}K'' - \frac{2}{r^{2}}
\end{aligned}$$

$$\begin{aligned}
\partial_{\alpha}\Gamma^{\alpha}_{11} &= \partial_{0}\Gamma^{0}_{11} + \partial_{1}\Gamma^{1}_{11} + \partial_{2}\Gamma^{2}_{11} + \partial_{3}\Gamma^{3}_{11} \\
&= 0 + \partial_{r}\left(\frac{1}{2}K'\right) + 0 + 0 \\
&= \frac{1}{2}K''
\end{aligned}$$

$$\begin{aligned}
\Gamma^{\alpha}_{\beta 1}\Gamma^{\beta}_{1\alpha} &= \Gamma^{0}_{\beta 1}\Gamma^{\beta}_{10} + \Gamma^{1}_{\beta 1}\Gamma^{\beta}_{11} + \Gamma^{2}_{\beta 1}\Gamma^{\beta}_{12} + \Gamma^{3}_{\beta 1}\Gamma^{\beta}_{13} \\
&= \Gamma^{0}_{01}\Gamma^{0}_{10} + \Gamma^{1}_{01}\Gamma^{0}_{11} + \Gamma^{2}_{01}\Gamma^{0}_{12} + \Gamma^{3}_{01}\Gamma^{0}_{13} + \\
&\quad \Gamma^{0}_{11}\Gamma^{1}_{10} + \Gamma^{1}_{11}\Gamma^{1}_{11} + \Gamma^{2}_{11}\Gamma^{1}_{12} + \Gamma^{3}_{11}\Gamma^{1}_{13} + \\
&\quad \Gamma^{0}_{21}\Gamma^{2}_{10} + \Gamma^{1}_{21}\Gamma^{2}_{11} + \Gamma^{2}_{21}\Gamma^{2}_{12} + \Gamma^{3}_{21}\Gamma^{2}_{13} + \\
&\quad \Gamma^{0}_{31}\Gamma^{3}_{10} + \Gamma^{1}_{31}\Gamma^{3}_{11} + \Gamma^{2}_{31}\Gamma^{3}_{12} + \Gamma^{3}_{31}\Gamma^{3}_{13} \\
&= \frac{1}{2}H'\frac{1}{2}H' + 0 + 0 + 0 + \\
&\quad 0 + \frac{1}{2}K'\frac{1}{2}K' + 0 + 0 + \\
&\quad 0 + 0 + \frac{1}{r}\frac{1}{r} + 0 +
\end{aligned}$$

$$
\begin{aligned}
& 0+0+0+\frac{1}{r}\frac{1}{r} \\
&= \frac{1}{4}\left(H'\right)^2+\frac{1}{4}\left(K'\right)^2+\frac{2}{r^2}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{11} &= \Gamma^{0}_{\beta 0}\Gamma^{\beta}_{11}+\Gamma^{1}_{\beta 1}\Gamma^{\beta}_{11}+\Gamma^{2}_{\beta 2}\Gamma^{\beta}_{11}+\Gamma^{3}_{\beta 3}\Gamma^{\beta}_{11} \\
&= \Gamma^{0}_{00}\Gamma^{0}_{11}+\Gamma^{1}_{01}\Gamma^{0}_{11}+\Gamma^{2}_{02}\Gamma^{0}_{11}+\Gamma^{3}_{03}\Gamma^{0}_{11}+ \\
&\quad \Gamma^{0}_{10}\Gamma^{1}_{11}+\Gamma^{1}_{11}\Gamma^{1}_{11}+\Gamma^{2}_{12}\Gamma^{1}_{11}+\Gamma^{3}_{13}\Gamma^{1}_{11}+ \\
&\quad \Gamma^{0}_{20}\Gamma^{2}_{11}+\Gamma^{1}_{21}\Gamma^{2}_{11}+\Gamma^{2}_{22}\Gamma^{2}_{11}+\Gamma^{3}_{23}\Gamma^{2}_{11}+ \\
&\quad \Gamma^{0}_{30}\Gamma^{3}_{11}+\Gamma^{1}_{31}\Gamma^{3}_{11}+\Gamma^{2}_{32}\Gamma^{3}_{11}+\Gamma^{3}_{33}\Gamma^{3}_{11} \\
&= 0+0+0+0+ \\
&\quad \frac{1}{2}H'\frac{1}{2}K'+\frac{1}{2}K'\frac{1}{2}K'+\frac{1}{r}\frac{1}{2}K'+\frac{1}{r}\frac{1}{2}K'+ \\
&\quad 0+0+0+0+ \\
&\quad 0+0+0+0 \\
&= \frac{1}{4}H'K'+\frac{1}{4}\left(K'\right)^2+\frac{K'}{r}
\end{aligned}
$$

$$
\begin{aligned}
R_{11} &= \partial_1\Gamma^{\alpha}_{1\alpha}-\partial_\alpha\Gamma^{\alpha}_{11}+\Gamma^{\alpha}_{\beta 1}\Gamma^{\beta}_{1\alpha}-\Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{11} \\
&= \frac{1}{2}H''+\frac{1}{2}K''-\frac{2}{r^2}-\frac{1}{2}K''+\frac{1}{4}\left(H'\right)^2+\frac{1}{4}\left(K'\right)^2+\frac{2}{r^2} \\
&\quad -\frac{1}{4}H'K'-\frac{1}{4}\left(K'\right)^2-\frac{K'}{r} \\
&= -\frac{H'K'}{4}+\frac{\left(H'\right)^2}{4}+\frac{H''}{2}-\frac{K'}{r}
\end{aligned}
$$

- $R_{22}=\partial_2\Gamma^{\alpha}_{2\alpha}-\partial_\alpha\Gamma^{\alpha}_{22}+\Gamma^{\alpha}_{\beta 2}\Gamma^{\beta}_{2\alpha}-\Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{22}$

$$
\begin{aligned}
\partial_2\Gamma^{\alpha}_{2\alpha} &= \partial_2\Gamma^{0}_{20}+\partial_2\Gamma^{1}_{21}+\partial_2\Gamma^{2}_{22}+\partial_2\Gamma^{3}_{23} \\
&= 0+0+0+\partial_\theta\cot\theta \\
&= -\csc^2\theta
\end{aligned}
$$

$$
\begin{aligned}
\partial_\alpha\Gamma^{\alpha}_{22} &= \partial_0\Gamma^{0}_{22}+\partial_1\Gamma^{1}_{22}+\partial_2\Gamma^{2}_{22}+\partial_3\Gamma^{3}_{22} \\
&= 0+\partial_r\left(-re^{-K}\right)+0+0 \\
&= -e^{-K}+re^{-K}K' \\
&= e^{-K}\left(rK'-1\right)
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^{\alpha}_{\beta 2}\Gamma^{\beta}_{2\alpha} &= \Gamma^{0}_{\beta 2}\Gamma^{\beta}_{20}+\Gamma^{1}_{\beta 2}\Gamma^{\beta}_{21}+\Gamma^{2}_{\beta 2}\Gamma^{\beta}_{22}+\Gamma^{3}_{\beta 2}\Gamma^{\beta}_{23} \\
&= \Gamma^{0}_{02}\Gamma^{0}_{20}+\Gamma^{1}_{02}\Gamma^{0}_{21}+\Gamma^{2}_{02}\Gamma^{0}_{22}+\Gamma^{3}_{02}\Gamma^{0}_{23}+ \\
&\quad \Gamma^{0}_{12}\Gamma^{1}_{20}+\Gamma^{1}_{12}\Gamma^{1}_{21}+\Gamma^{2}_{12}\Gamma^{1}_{22}+\Gamma^{3}_{12}\Gamma^{1}_{23}+ \\
&\quad \Gamma^{0}_{22}\Gamma^{2}_{20}+\Gamma^{1}_{22}\Gamma^{2}_{21}+\Gamma^{2}_{22}\Gamma^{2}_{22}+\Gamma^{3}_{22}\Gamma^{2}_{23}+ \\
&\quad \Gamma^{0}_{32}\Gamma^{3}_{20}+\Gamma^{1}_{32}\Gamma^{3}_{21}+\Gamma^{2}_{32}\Gamma^{3}_{22}+\Gamma^{3}_{32}\Gamma^{3}_{23} \\
&= 0+0+0+0+ \\
&\quad 0+0-\frac{1}{r}re^{-K}+0+ \\
&\quad 0-re^{-K}\frac{1}{r}+0+0+
\end{aligned}
$$

$$
\begin{aligned}
& 0+0+0+\cot\theta\cot\theta \\
= {} & \cot^2\theta - 2e^{-K}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{22} &= \Gamma^{0}_{\beta 0}\Gamma^{\beta}_{22}+\Gamma^{1}_{\beta 1}\Gamma^{\beta}_{22}+\Gamma^{2}_{\beta 2}\Gamma^{\beta}_{22}+\Gamma^{3}_{\beta 3}\Gamma^{\beta}_{22} \\
&= \Gamma^{0}_{00}\Gamma^{0}_{22}+\Gamma^{1}_{01}\Gamma^{0}_{22}+\Gamma^{2}_{02}\Gamma^{0}_{22}+\Gamma^{3}_{03}\Gamma^{0}_{22}+ \\
&\quad \Gamma^{0}_{10}\Gamma^{1}_{22}+\Gamma^{1}_{11}\Gamma^{1}_{22}+\Gamma^{2}_{12}\Gamma^{1}_{22}+\Gamma^{3}_{13}\Gamma^{1}_{22}+ \\
&\quad \Gamma^{0}_{20}\Gamma^{2}_{22}+\Gamma^{1}_{21}\Gamma^{2}_{22}+\Gamma^{2}_{22}\Gamma^{2}_{22}+\Gamma^{3}_{23}\Gamma^{2}_{22}+ \\
&\quad \Gamma^{0}_{30}\Gamma^{3}_{22}+\Gamma^{1}_{31}\Gamma^{3}_{22}+\Gamma^{2}_{32}\Gamma^{3}_{22}+\Gamma^{3}_{33}\Gamma^{3}_{22} \\
&= 0+0+0+0+ \\
&\quad -\frac{1}{2}H're^{-K}-\frac{1}{2}K're^{-K}-\frac{1}{r}re^{-K}-\frac{1}{r}re^{-K}+ \\
&\quad 0+0+0+0+ \\
&\quad 0+0+0+0 \\
&= e^{-K}\left(-\frac{r}{2}H'-\frac{r}{2}K'-2\right)
\end{aligned}
$$

$$
\begin{aligned}
R_{22} &= \partial_2\Gamma^{\alpha}_{2\alpha}-\partial_\alpha\Gamma^{\alpha}_{22}+\Gamma^{\alpha}_{\beta 2}\Gamma^{\beta}_{2\alpha}-\Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{22} \\
&= -\csc^2\theta-e^{-K}\left(rK'-1\right)+\cot^2\theta-2e^{-K}+e^{-K}\left(\frac{r}{2}H'+\frac{r}{2}K'+2\right) \\
&= e^{-K}\left[1+\frac{r\left(H'-K'\right)}{2}\right]-\left(\csc^2\theta-\cot^2\theta\right) \\
&= e^{-K}\left[1+\frac{r\left(H'-K'\right)}{2}\right]-1
\end{aligned}
$$

- $R_{33}=\partial_3\Gamma^{\alpha}_{3\alpha}-\partial_\alpha\Gamma^{\alpha}_{33}+\Gamma^{\alpha}_{\beta 3}\Gamma^{\beta}_{3\alpha}-\Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{33}$

$$
\begin{aligned}
\partial_3\Gamma^{\alpha}_{3\alpha} &= \partial_3\Gamma^{0}_{30}+\partial_3\Gamma^{1}_{31}+\partial_3\Gamma^{2}_{32}+\partial_3\Gamma^{3}_{33} \\
&= 0+0+0+0 \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\partial_\alpha\Gamma^{\alpha}_{33} &= \partial_0\Gamma^{0}_{33}+\partial_1\Gamma^{1}_{33}+\partial_2\Gamma^{2}_{33}+\partial_3\Gamma^{3}_{33} \\
&= 0+\partial_r\left(-re^{-K}\sin^2\theta\right)+\partial_\theta\left(-\sin\theta\cos\theta\right)+0 \\
&= -e^{-K}\sin^2\theta+rK'e^{-K}\sin^2\theta-\cos^2\theta+\sin^2\theta
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^{\alpha}_{\beta 3}\Gamma^{\beta}_{3\alpha} &= \Gamma^{0}_{\beta 3}\Gamma^{\beta}_{30}+\Gamma^{1}_{\beta 3}\Gamma^{\beta}_{31}+\Gamma^{2}_{\beta 3}\Gamma^{\beta}_{32}+\Gamma^{3}_{\beta 3}\Gamma^{\beta}_{33} \\
&= \Gamma^{0}_{03}\Gamma^{0}_{30}+\Gamma^{1}_{03}\Gamma^{0}_{31}+\Gamma^{2}_{03}\Gamma^{0}_{32}+\Gamma^{3}_{03}\Gamma^{0}_{33}+ \\
&\quad \Gamma^{0}_{13}\Gamma^{1}_{30}+\Gamma^{1}_{13}\Gamma^{1}_{31}+\Gamma^{2}_{13}\Gamma^{1}_{32}+\Gamma^{3}_{13}\Gamma^{1}_{33}+ \\
&\quad \Gamma^{0}_{23}\Gamma^{2}_{30}+\Gamma^{1}_{23}\Gamma^{2}_{31}+\Gamma^{2}_{23}\Gamma^{2}_{32}+\Gamma^{3}_{23}\Gamma^{2}_{33}+ \\
&\quad \Gamma^{0}_{33}\Gamma^{3}_{30}+\Gamma^{1}_{33}\Gamma^{3}_{31}+\Gamma^{2}_{33}\Gamma^{3}_{32}+\Gamma^{3}_{33}\Gamma^{3}_{33} \\
&= 0+0+0+0+ \\
&\quad 0+0+0-\frac{1}{r}re^{-K}\sin^2\theta+ \\
&\quad 0+0+0-\cot\theta\sin\theta\cos\theta+ \\
&\quad 0-re^{-K}\sin^2\theta\frac{1}{r}-\sin\theta\cos\theta\cot\theta+0 \\
&= -2e^{-K}\sin^2\theta-2\cos^2\theta
\end{aligned}
$$

$$\begin{aligned}
\Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{33} &= \Gamma^{0}_{\beta 0}\Gamma^{\beta}_{33} + \Gamma^{1}_{\beta 1}\Gamma^{\beta}_{33} + \Gamma^{2}_{\beta 2}\Gamma^{\beta}_{33} + \Gamma^{3}_{\beta 3}\Gamma^{\beta}_{33} \\
&= \Gamma^{0}_{00}\Gamma^{0}_{33} + \Gamma^{1}_{01}\Gamma^{0}_{33} + \Gamma^{2}_{02}\Gamma^{0}_{33} + \Gamma^{3}_{03}\Gamma^{0}_{33} + \\
&\quad \Gamma^{0}_{10}\Gamma^{1}_{33} + \Gamma^{1}_{11}\Gamma^{1}_{33} + \Gamma^{2}_{12}\Gamma^{1}_{33} + \Gamma^{3}_{13}\Gamma^{1}_{33} + \\
&\quad \Gamma^{0}_{20}\Gamma^{2}_{33} + \Gamma^{1}_{21}\Gamma^{2}_{33} + \Gamma^{2}_{22}\Gamma^{2}_{33} + \Gamma^{3}_{23}\Gamma^{2}_{33} + \\
&\quad \Gamma^{0}_{30}\Gamma^{3}_{33} + \Gamma^{1}_{31}\Gamma^{3}_{33} + \Gamma^{2}_{32}\Gamma^{3}_{33} + \Gamma^{3}_{33}\Gamma^{3}_{33} \\
&= 0+0+0+0 \\
&\quad -\frac{1}{2}H' r e^{-K}\sin^2\theta - \frac{1}{2}K' r e^{-K}\sin^2\theta - \frac{1}{r} r e^{-K}\sin^2\theta - \frac{1}{r} r e^{-K}\sin^2\theta + \\
&\quad 0+0+0-\cot\theta\sin\theta\cos\theta + \\
&\quad 0+0+0+0 \\
&= -\frac{1}{2} r e^{-K}\sin^2\theta\,(H'+K') - 2e^{-K}\sin^2\theta - \cos^2\theta
\end{aligned}$$

$$\begin{aligned}
R_{33} &= \partial_3\Gamma^{\alpha}_{3\alpha} - \partial_\alpha\Gamma^{\alpha}_{33} + \Gamma^{\alpha}_{\beta 3}\Gamma^{\beta}_{3\alpha} - \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{33} \\
&= 0 + e^{-K}\sin^2\theta - rK'e^{-K}\sin^2\theta + \cos^2\theta - \sin^2\theta - 2e^{-K}\sin^2\theta - 2\cos^2\theta \\
&\quad + \frac{1}{2} r e^{-K}\sin^2\theta\,(H'+K') + 2e^{-K}\sin^2\theta + \cos^2\theta \\
&= \sin^2\theta\left\{e^{-K}\left[1 + \frac{r\,(H'-K')}{2}\right] - 1\right\}
\end{aligned}$$

- $R_{01} = R_{10} = \partial_1\Gamma^{\alpha}_{0\alpha} - \partial_\alpha\Gamma^{\alpha}_{01} + \Gamma^{\alpha}_{\beta 1}\Gamma^{\beta}_{0\alpha} - \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{01}$

$$\begin{aligned}
\partial_1\Gamma^{\alpha}_{0\alpha} &= \partial_1\Gamma^{0}_{00} + \partial_1\Gamma^{1}_{01} + \partial_1\Gamma^{2}_{02} + \partial_1\Gamma^{3}_{03} \\
&= 0+0+0+0 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\partial_\alpha\Gamma^{\alpha}_{01} &= \partial_0\Gamma^{0}_{01} + \partial_1\Gamma^{1}_{01} + \partial_2\Gamma^{2}_{01} + \partial_3\Gamma^{3}_{01} \\
&= \partial_t\left(\frac{1}{2}H'\right) + 0+0+0 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\Gamma^{\alpha}_{\beta 1}\Gamma^{\beta}_{0\alpha} &= \Gamma^{0}_{\beta 1}\Gamma^{\beta}_{00} + \Gamma^{1}_{\beta 1}\Gamma^{\beta}_{01} + \Gamma^{2}_{\beta 1}\Gamma^{\beta}_{02} + \Gamma^{3}_{\beta 1}\Gamma^{\beta}_{03} \\
&= \Gamma^{0}_{01}\Gamma^{0}_{00} + \Gamma^{1}_{01}\Gamma^{0}_{01} + \Gamma^{2}_{01}\Gamma^{0}_{02} + \Gamma^{3}_{01}\Gamma^{0}_{03} + \\
&\quad \Gamma^{0}_{11}\Gamma^{1}_{00} + \Gamma^{1}_{11}\Gamma^{1}_{01} + \Gamma^{2}_{11}\Gamma^{1}_{02} + \Gamma^{3}_{11}\Gamma^{1}_{03} + \\
&\quad \Gamma^{0}_{21}\Gamma^{2}_{00} + \Gamma^{1}_{21}\Gamma^{2}_{01} + \Gamma^{2}_{21}\Gamma^{2}_{02} + \Gamma^{3}_{21}\Gamma^{2}_{03} + \\
&\quad \Gamma^{0}_{31}\Gamma^{3}_{00} + \Gamma^{1}_{31}\Gamma^{3}_{01} + \Gamma^{2}_{31}\Gamma^{3}_{02} + \Gamma^{3}_{31}\Gamma^{3}_{03} \\
&= 0+0+0+0+ \\
&\quad 0+0+0+0+ \\
&\quad 0+0+0+0+ \\
&\quad 0+0+0+0 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{01} &= \Gamma^{0}_{\beta 0}\Gamma^{\beta}_{01} + \Gamma^{1}_{\beta 1}\Gamma^{\beta}_{01} + \Gamma^{2}_{\beta 2}\Gamma^{\beta}_{01} + \Gamma^{3}_{\beta 3}\Gamma^{\beta}_{01} \\
&= \Gamma^{0}_{00}\Gamma^{0}_{01} + \Gamma^{1}_{01}\Gamma^{0}_{01} + \Gamma^{2}_{02}\Gamma^{0}_{01} + \Gamma^{3}_{03}\Gamma^{0}_{01} +
\end{aligned}$$

$$\begin{aligned}
& \Gamma^0_{10}\Gamma^1_{01} + \Gamma^1_{11}\Gamma^1_{01} + \Gamma^2_{12}\Gamma^1_{01} + \Gamma^3_{13}\Gamma^1_{01} + \\
& \Gamma^0_{20}\Gamma^2_{01} + \Gamma^1_{21}\Gamma^2_{01} + \Gamma^2_{22}\Gamma^2_{01} + \Gamma^3_{23}\Gamma^2_{01} + \\
& \Gamma^0_{30}\Gamma^3_{01} + \Gamma^1_{31}\Gamma^3_{01} + \Gamma^2_{32}\Gamma^3_{01} + \Gamma^3_{33}\Gamma^3_{01} \\
= \;& 0+0+0+0+ \\
& 0+0+0+0+ \\
& 0+0+0+0+ \\
& 0+0+0+0 \\
= \;& 0
\end{aligned}$$

$$R_{01} = R_{10} = 0 - 0 + 0 - 0 = 0$$

- $R_{02} = R_{20} = \partial_2\Gamma^\alpha_{0\alpha} - \partial_\alpha\Gamma^\alpha_{02} + \Gamma^\alpha_{\beta 2}\Gamma^\beta_{0\alpha} - \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{02}$

$$\begin{aligned}
\partial_2\Gamma^\alpha_{0\alpha} &= \partial_2\Gamma^0_{00} + \partial_2\Gamma^1_{01} + \partial_2\Gamma^2_{02} + \partial_2\Gamma^3_{03} \\
&= 0+0+0+0 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\partial_\alpha\Gamma^\alpha_{02} &= \partial_0\Gamma^0_{02} + \partial_1\Gamma^1_{02} + \partial_2\Gamma^2_{02} + \partial_3\Gamma^3_{02} \\
&= 0+0+0+0 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\Gamma^\alpha_{\beta 2}\Gamma^\beta_{0\alpha} &= \Gamma^0_{\beta 2}\Gamma^\beta_{00} + \Gamma^1_{\beta 2}\Gamma^\beta_{01} + \Gamma^2_{\beta 2}\Gamma^\beta_{02} + \Gamma^3_{\beta 2}\Gamma^\beta_{03} \\
&= \Gamma^0_{02}\Gamma^0_{00} + \Gamma^1_{02}\Gamma^0_{01} + \Gamma^2_{02}\Gamma^0_{02} + \Gamma^3_{02}\Gamma^0_{03} + \\
&\quad \Gamma^0_{12}\Gamma^1_{00} + \Gamma^1_{12}\Gamma^1_{01} + \Gamma^2_{12}\Gamma^1_{02} + \Gamma^3_{12}\Gamma^1_{03} + \\
&\quad \Gamma^0_{22}\Gamma^2_{00} + \Gamma^1_{22}\Gamma^2_{01} + \Gamma^2_{22}\Gamma^2_{02} + \Gamma^3_{22}\Gamma^2_{03} + \\
&\quad \Gamma^0_{32}\Gamma^3_{00} + \Gamma^1_{32}\Gamma^3_{01} + \Gamma^2_{32}\Gamma^3_{02} + \Gamma^3_{32}\Gamma^3_{03} \\
&= 0+0+0+0+ \\
&\quad 0+0+0+0+ \\
&\quad 0+0+0+0+ \\
&\quad 0+0+0+0 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{02} &= \Gamma^0_{\beta 0}\Gamma^\beta_{02} + \Gamma^1_{\beta 1}\Gamma^\beta_{02} + \Gamma^2_{\beta 2}\Gamma^\beta_{02} + \Gamma^3_{\beta 3}\Gamma^\beta_{02} \\
&= \Gamma^0_{00}\Gamma^0_{02} + \Gamma^1_{01}\Gamma^0_{02} + \Gamma^2_{02}\Gamma^0_{02} + \Gamma^3_{03}\Gamma^0_{02} + \\
&\quad \Gamma^0_{10}\Gamma^1_{02} + \Gamma^1_{11}\Gamma^1_{02} + \Gamma^2_{12}\Gamma^1_{02} + \Gamma^3_{13}\Gamma^1_{02} + \\
&\quad \Gamma^0_{20}\Gamma^2_{02} + \Gamma^1_{21}\Gamma^2_{02} + \Gamma^2_{22}\Gamma^2_{02} + \Gamma^3_{23}\Gamma^2_{02} + \\
&\quad \Gamma^0_{30}\Gamma^3_{02} + \Gamma^1_{31}\Gamma^3_{02} + \Gamma^2_{32}\Gamma^3_{02} + \Gamma^3_{33}\Gamma^3_{02} \\
&= 0+0+0+0+ \\
&\quad 0+0+0+0+ \\
&\quad 0+0+0+0+ \\
&\quad 0+0+0+0 \\
&= 0
\end{aligned}$$

$$R_{02} = R_{20} = 0 - 0 + 0 - 0 = 0$$

- $R_{03} = R_{30} = \partial_3 \Gamma^{\alpha}_{0\alpha} - \partial_\alpha \Gamma^{\alpha}_{03} + \Gamma^{\alpha}_{\beta 3}\Gamma^{\beta}_{0\alpha} - \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{03}$

$$\begin{aligned}
\partial_3 \Gamma^{\alpha}_{0\alpha} &= \partial_3 \Gamma^{0}_{00} + \partial_3 \Gamma^{1}_{01} + \partial_3 \Gamma^{2}_{02} + \partial_3 \Gamma^{3}_{03} \\
&= 0 + 0 + 0 + 0 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\partial_\alpha \Gamma^{\alpha}_{03} &= \partial_0 \Gamma^{0}_{03} + \partial_1 \Gamma^{1}_{03} + \partial_2 \Gamma^{2}_{03} + \partial_3 \Gamma^{3}_{03} \\
&= 0 + 0 + 0 + 0 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\Gamma^{\alpha}_{\beta 3}\Gamma^{\beta}_{0\alpha} &= \Gamma^{0}_{\beta 3}\Gamma^{\beta}_{00} + \Gamma^{1}_{\beta 3}\Gamma^{\beta}_{01} + \Gamma^{2}_{\beta 3}\Gamma^{\beta}_{02} + \Gamma^{3}_{\beta 3}\Gamma^{\beta}_{03} \\
&= \Gamma^{0}_{03}\Gamma^{0}_{00} + \Gamma^{1}_{03}\Gamma^{0}_{01} + \Gamma^{2}_{03}\Gamma^{0}_{02} + \Gamma^{3}_{03}\Gamma^{0}_{03} + \\
&\quad \Gamma^{0}_{13}\Gamma^{1}_{00} + \Gamma^{1}_{13}\Gamma^{1}_{01} + \Gamma^{2}_{13}\Gamma^{1}_{02} + \Gamma^{3}_{13}\Gamma^{1}_{03} + \\
&\quad \Gamma^{0}_{23}\Gamma^{2}_{00} + \Gamma^{1}_{23}\Gamma^{2}_{01} + \Gamma^{2}_{23}\Gamma^{2}_{02} + \Gamma^{3}_{23}\Gamma^{2}_{03} + \\
&\quad \Gamma^{0}_{33}\Gamma^{3}_{00} + \Gamma^{1}_{33}\Gamma^{3}_{01} + \Gamma^{2}_{33}\Gamma^{3}_{02} + \Gamma^{3}_{33}\Gamma^{3}_{03} \\
&= 0 + 0 + 0 + 0 + \\
&\quad 0 + 0 + 0 + 0 + \\
&\quad 0 + 0 + 0 + 0 + \\
&\quad 0 + 0 + 0 + 0 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{03} &= \Gamma^{0}_{\beta 0}\Gamma^{\beta}_{03} + \Gamma^{1}_{\beta 1}\Gamma^{\beta}_{03} + \Gamma^{2}_{\beta 2}\Gamma^{\beta}_{03} + \Gamma^{3}_{\beta 3}\Gamma^{\beta}_{03} \\
&= \Gamma^{0}_{00}\Gamma^{0}_{03} + \Gamma^{1}_{01}\Gamma^{0}_{03} + \Gamma^{2}_{02}\Gamma^{0}_{03} + \Gamma^{3}_{03}\Gamma^{0}_{03} + \\
&\quad \Gamma^{0}_{10}\Gamma^{1}_{03} + \Gamma^{1}_{11}\Gamma^{1}_{03} + \Gamma^{2}_{12}\Gamma^{1}_{03} + \Gamma^{3}_{13}\Gamma^{1}_{03} + \\
&\quad \Gamma^{0}_{20}\Gamma^{2}_{03} + \Gamma^{1}_{21}\Gamma^{2}_{03} + \Gamma^{2}_{22}\Gamma^{2}_{03} + \Gamma^{3}_{23}\Gamma^{2}_{03} + \\
&\quad \Gamma^{0}_{30}\Gamma^{3}_{03} + \Gamma^{1}_{31}\Gamma^{3}_{03} + \Gamma^{2}_{32}\Gamma^{3}_{03} + \Gamma^{3}_{33}\Gamma^{3}_{03} \\
&= 0 + 0 + 0 + 0 + \\
&\quad 0 + 0 + 0 + 0 + \\
&\quad 0 + 0 + 0 + 0 + \\
&\quad 0 + 0 + 0 + 0 \\
&= 0
\end{aligned}$$

$$R_{03} = R_{30} = 0 - 0 + 0 - 0 = 0$$

- $R_{12} = R_{21} = \partial_2 \Gamma^{\alpha}_{1\alpha} - \partial_\alpha \Gamma^{\alpha}_{12} + \Gamma^{\alpha}_{\beta 2}\Gamma^{\beta}_{1\alpha} - \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{12}$

$$\begin{aligned}
\partial_2 \Gamma^{\alpha}_{1\alpha} &= \partial_2 \Gamma^{0}_{10} + \partial_2 \Gamma^{1}_{11} + \partial_2 \Gamma^{2}_{12} + \partial_2 \Gamma^{3}_{13} \\
&= \partial_\theta \left(\frac{1}{2}H'\right) + \partial_\theta \left(\frac{1}{2}K'\right) + \partial_\theta \left(\frac{1}{r}\right) + \partial_\theta \left(\frac{1}{r}\right) \\
&= 0 + 0 + 0 + 0 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\partial_\alpha \Gamma^\alpha_{12} &= \partial_0\Gamma^0_{12} + \partial_1\Gamma^1_{12} + \partial_2\Gamma^2_{12} + \partial_3\Gamma^3_{12} \\
&= 0 + 0 + \partial_\theta\left(\frac{1}{r}\right) + 0 \\
&= 0 + 0 + 0 + 0 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\Gamma^\alpha_{\beta 2}\Gamma^\beta_{1\alpha} &= \Gamma^0_{\beta 2}\Gamma^\beta_{10} + \Gamma^1_{\beta 2}\Gamma^\beta_{11} + \Gamma^2_{\beta 2}\Gamma^\beta_{12} + \Gamma^3_{\beta 2}\Gamma^\beta_{13} \\
&= \Gamma^0_{02}\Gamma^0_{10} + \Gamma^1_{02}\Gamma^0_{11} + \Gamma^2_{02}\Gamma^0_{12} + \Gamma^3_{02}\Gamma^0_{13} + \\
&\quad \Gamma^0_{12}\Gamma^1_{10} + \Gamma^1_{12}\Gamma^1_{11} + \Gamma^2_{12}\Gamma^1_{12} + \Gamma^3_{12}\Gamma^1_{13} + \\
&\quad \Gamma^0_{22}\Gamma^2_{10} + \Gamma^1_{22}\Gamma^2_{11} + \Gamma^2_{22}\Gamma^2_{12} + \Gamma^3_{22}\Gamma^2_{13} + \\
&\quad \Gamma^0_{32}\Gamma^3_{10} + \Gamma^1_{32}\Gamma^3_{11} + \Gamma^2_{32}\Gamma^3_{12} + \Gamma^3_{32}\Gamma^3_{13} \\
&= 0 + 0 + 0 + 0 + \\
&\quad 0 + 0 + 0 + 0 + \\
&\quad 0 + 0 + 0 + 0 + \\
&\quad 0 + 0 + 0 + \frac{\cot\theta}{r} \\
&= \frac{\cot\theta}{r}
\end{aligned}$$

$$\begin{aligned}
\Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{12} &= \Gamma^0_{\beta 0}\Gamma^\beta_{12} + \Gamma^1_{\beta 1}\Gamma^\beta_{12} + \Gamma^2_{\beta 2}\Gamma^\beta_{12} + \Gamma^3_{\beta 3}\Gamma^\beta_{12} \\
&= \Gamma^0_{00}\Gamma^0_{12} + \Gamma^1_{01}\Gamma^0_{12} + \Gamma^2_{02}\Gamma^0_{12} + \Gamma^3_{03}\Gamma^0_{12} + \\
&\quad \Gamma^0_{10}\Gamma^1_{12} + \Gamma^1_{11}\Gamma^1_{12} + \Gamma^2_{12}\Gamma^1_{12} + \Gamma^3_{13}\Gamma^1_{12} + \\
&\quad \Gamma^0_{20}\Gamma^2_{12} + \Gamma^1_{21}\Gamma^2_{12} + \Gamma^2_{22}\Gamma^2_{12} + \Gamma^3_{23}\Gamma^2_{12} + \\
&\quad \Gamma^0_{30}\Gamma^3_{12} + \Gamma^1_{31}\Gamma^3_{12} + \Gamma^2_{32}\Gamma^3_{12} + \Gamma^3_{33}\Gamma^3_{12} \\
&= 0 + 0 + 0 + 0 + \\
&\quad 0 + 0 + 0 + 0 + \\
&\quad 0 + 0 + 0 + \frac{\cot\theta}{r} + \\
&\quad 0 + 0 + 0 + 0 \\
&= \frac{\cot\theta}{r}
\end{aligned}$$

$$R_{12} = R_{21} = 0 - 0 + \frac{\cot\theta}{r} - \frac{\cot\theta}{r} = 0$$

- $R_{13} = R_{31} = \partial_3\Gamma^\alpha_{1\alpha} - \partial_\alpha\Gamma^\alpha_{13} + \Gamma^\alpha_{\beta 3}\Gamma^\beta_{1\alpha} - \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{13}$

$$\begin{aligned}
\partial_3\Gamma^\alpha_{1\alpha} &= \partial_3\Gamma^0_{10} + \partial_3\Gamma^1_{11} + \partial_3\Gamma^2_{12} + \partial_3\Gamma^3_{13} \\
&= \partial_\phi\left(\frac{1}{2}H'\right) + \partial_\phi\left(\frac{1}{2}K'\right) + \partial_\phi\left(\frac{1}{r}\right) + \partial_\phi\left(\frac{1}{r}\right) \\
&= 0 + 0 + 0 + 0 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\partial_\alpha\Gamma^\alpha_{13} &= \partial_0\Gamma^0_{13} + \partial_1\Gamma^1_{13} + \partial_2\Gamma^2_{13} + \partial_3\Gamma^3_{13} \\
&= 0 + 0 + 0 + \partial_\phi\left(\frac{1}{r}\right) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\Gamma^{\alpha}_{\beta 3}\Gamma^{\beta}_{1\alpha} &= \Gamma^{0}_{\beta 3}\Gamma^{\beta}_{10} + \Gamma^{1}_{\beta 3}\Gamma^{\beta}_{11} + \Gamma^{2}_{\beta 3}\Gamma^{\beta}_{12} + \Gamma^{3}_{\beta 3}\Gamma^{\beta}_{13} \\
&= \Gamma^{0}_{03}\Gamma^{0}_{10} + \Gamma^{1}_{03}\Gamma^{0}_{11} + \Gamma^{2}_{03}\Gamma^{0}_{12} + \Gamma^{3}_{03}\Gamma^{0}_{13} + \\
&\quad \Gamma^{0}_{13}\Gamma^{1}_{10} + \Gamma^{1}_{13}\Gamma^{1}_{11} + \Gamma^{2}_{13}\Gamma^{1}_{12} + \Gamma^{3}_{13}\Gamma^{1}_{13} + \\
&\quad \Gamma^{0}_{23}\Gamma^{2}_{10} + \Gamma^{1}_{23}\Gamma^{2}_{11} + \Gamma^{2}_{23}\Gamma^{2}_{12} + \Gamma^{3}_{23}\Gamma^{2}_{13} + \\
&\quad \Gamma^{0}_{33}\Gamma^{3}_{10} + \Gamma^{1}_{33}\Gamma^{3}_{11} + \Gamma^{2}_{33}\Gamma^{3}_{12} + \Gamma^{3}_{33}\Gamma^{3}_{13} \\
&= 0 + 0 + 0 + 0 + \\
&\quad 0 + 0 + 0 + 0 + \\
&\quad 0 + 0 + 0 + 0 + \\
&\quad 0 + 0 + 0 + 0 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{13} &= \Gamma^{0}_{\beta 0}\Gamma^{\beta}_{13} + \Gamma^{1}_{\beta 1}\Gamma^{\beta}_{13} + \Gamma^{2}_{\beta 2}\Gamma^{\beta}_{13} + \Gamma^{3}_{\beta 3}\Gamma^{\beta}_{13} \\
&= \Gamma^{0}_{00}\Gamma^{0}_{13} + \Gamma^{1}_{01}\Gamma^{0}_{13} + \Gamma^{2}_{02}\Gamma^{0}_{13} + \Gamma^{3}_{03}\Gamma^{0}_{13} + \\
&\quad \Gamma^{0}_{10}\Gamma^{1}_{13} + \Gamma^{1}_{11}\Gamma^{1}_{13} + \Gamma^{2}_{12}\Gamma^{1}_{13} + \Gamma^{3}_{13}\Gamma^{1}_{13} + \\
&\quad \Gamma^{0}_{20}\Gamma^{2}_{13} + \Gamma^{1}_{21}\Gamma^{2}_{13} + \Gamma^{2}_{22}\Gamma^{2}_{13} + \Gamma^{3}_{23}\Gamma^{2}_{13} + \\
&\quad \Gamma^{0}_{30}\Gamma^{3}_{13} + \Gamma^{1}_{31}\Gamma^{3}_{13} + \Gamma^{2}_{32}\Gamma^{3}_{13} + \Gamma^{3}_{33}\Gamma^{3}_{13} \\
&= 0 + 0 + 0 + 0 + \\
&\quad 0 + 0 + 0 + 0 + \\
&\quad 0 + 0 + 0 + 0 + \\
&\quad 0 + 0 + 0 + 0 \\
&= 0
\end{aligned}$$

$$R_{13} = R_{31} = 0 - 0 + 0 - 0 = 0$$

- $R_{23} = R_{32} = \partial_3\Gamma^{\alpha}_{2\alpha} - \partial_\alpha\Gamma^{\alpha}_{23} + \Gamma^{\alpha}_{\beta 3}\Gamma^{\beta}_{2\alpha} - \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{23}$

$$\begin{aligned}
\partial_3\Gamma^{\alpha}_{2\alpha} &= \partial_3\Gamma^{0}_{20} + \partial_3\Gamma^{1}_{21} + \partial_3\Gamma^{2}_{22} + \partial_3\Gamma^{3}_{23} \\
&= 0 + 0 + 0 + \partial_\phi\left(\cot\theta\right) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\partial_\alpha\Gamma^{\alpha}_{23} &= \partial_0\Gamma^{0}_{23} + \partial_1\Gamma^{1}_{23} + \partial_2\Gamma^{2}_{23} + \partial_3\Gamma^{3}_{23} \\
&= 0 + 0 + 0 + \partial_\phi\left(\cot\theta\right) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\Gamma^{\alpha}_{\beta 3}\Gamma^{\beta}_{2\alpha} &= \Gamma^{0}_{\beta 3}\Gamma^{\beta}_{20} + \Gamma^{1}_{\beta 3}\Gamma^{\beta}_{21} + \Gamma^{2}_{\beta 3}\Gamma^{\beta}_{22} + \Gamma^{3}_{\beta 3}\Gamma^{\beta}_{23} \\
&= \Gamma^{0}_{03}\Gamma^{0}_{20} + \Gamma^{1}_{03}\Gamma^{0}_{21} + \Gamma^{2}_{03}\Gamma^{0}_{22} + \Gamma^{3}_{03}\Gamma^{0}_{23} + \\
&\quad \Gamma^{0}_{13}\Gamma^{1}_{20} + \Gamma^{1}_{13}\Gamma^{1}_{21} + \Gamma^{2}_{13}\Gamma^{1}_{22} + \Gamma^{3}_{13}\Gamma^{1}_{23} + \\
&\quad \Gamma^{0}_{23}\Gamma^{2}_{20} + \Gamma^{1}_{23}\Gamma^{2}_{21} + \Gamma^{2}_{23}\Gamma^{2}_{22} + \Gamma^{3}_{23}\Gamma^{2}_{23} + \\
&\quad \Gamma^{0}_{33}\Gamma^{3}_{20} + \Gamma^{1}_{33}\Gamma^{3}_{21} + \Gamma^{2}_{33}\Gamma^{3}_{22} + \Gamma^{3}_{33}\Gamma^{3}_{23} \\
&= 0 + 0 + 0 + 0 + \\
&\quad 0 + 0 + 0 + 0 + \\
&\quad 0 + 0 + 0 + 0 +
\end{aligned}$$

$$
\begin{aligned}
& && 0+0+0+0 \\
& = && 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{23} & = && \Gamma^{0}_{\beta 0}\Gamma^{\beta}_{23}+\Gamma^{1}_{\beta 1}\Gamma^{\beta}_{23}+\Gamma^{2}_{\beta 2}\Gamma^{\beta}_{23}+\Gamma^{3}_{\beta 3}\Gamma^{\beta}_{23} \\
& = && \Gamma^{0}_{00}\Gamma^{0}_{23}+\Gamma^{1}_{01}\Gamma^{0}_{23}+\Gamma^{2}_{02}\Gamma^{0}_{23}+\Gamma^{3}_{03}\Gamma^{0}_{23}+ \\
& && \Gamma^{0}_{10}\Gamma^{1}_{23}+\Gamma^{1}_{11}\Gamma^{1}_{23}+\Gamma^{2}_{12}\Gamma^{1}_{23}+\Gamma^{3}_{13}\Gamma^{1}_{23}+ \\
& && \Gamma^{0}_{20}\Gamma^{2}_{23}+\Gamma^{1}_{21}\Gamma^{2}_{23}+\Gamma^{2}_{22}\Gamma^{2}_{23}+\Gamma^{3}_{23}\Gamma^{2}_{23}+ \\
& && \Gamma^{0}_{30}\Gamma^{3}_{23}+\Gamma^{1}_{31}\Gamma^{3}_{23}+\Gamma^{2}_{32}\Gamma^{3}_{23}+\Gamma^{3}_{33}\Gamma^{3}_{23} \\
& = && 0+0+0+0+ \\
& && 0+0+0+0+ \\
& && 0+0+0+0+ \\
& && 0+0+0+0 \\
& = && 0
\end{aligned}
$$

$$
R_{23} = R_{32} = 0-0+0-0 = 0
$$

13. Show that the differential equation $e^{H}\left(1+rH'\right)=1$ has the following solution: $e^{H}=1-\frac{C_s}{r}$ where C_s is a constant and the prime represents derivative with respect to r.
Answer: If $e^{H}\equiv y$ then $\frac{dy}{dr}=e^{H}\frac{dH}{dr}=yH'$ and we have:

$$
\begin{aligned}
e^{H}\left(1+rH'\right) &= 1 \\
y\left(1+rH'\right) &= 1 \\
y+ryH' &= 1 \\
y+r\frac{dy}{dr} &= 1 \\
\frac{d}{dr}\left(ry\right) &= 1 \\
ry &= r+C \\
y &= 1+\frac{C}{r} \\
e^{H} &= 1+\frac{C}{r} \\
e^{H} &= 1-\frac{C_s}{r}
\end{aligned}
$$

where C is the constant of integration and $C_s=-C$.
Note: for verification, we have:

$$
e^{H}=1-\frac{C_s}{r} \qquad H=\ln\left(1-\frac{C_s}{r}\right) \qquad H'=\left(1-\frac{C_s}{r}\right)^{-1}\frac{C_s}{r^2}
$$

and hence:

$$
e^{H}\left(1+rH'\right)=\left(1-\frac{C_s}{r}\right)\left(1+r\left[\left(1-\frac{C_s}{r}\right)^{-1}\frac{C_s}{r^2}\right]\right)=\left(1-\frac{C_s}{r}\right)+\frac{C_s}{r}=1
$$

14. What is Schwarzschild radius?
Answer: Schwarzschild radius (which may also be called gravitational radius and is usually associated with black holes although it is not specific to them) is the distance from the center of the gravitating

object to the points of space where the escape speed is equal to the characteristic speed of light c (assuming that the mass of the object is confined to the region within the Schwarzschild radius). We should note that this definition is essentially classical. In fact, the general relativistic definition of the Schwarzschild radius is obtained from the Schwarzschild metric and the singularity at $r = \frac{2GM}{c^2}$ (or by comparing the classical and general relativistic formulations in the classical limit, as will be discussed in § 5.1). Anyway, the classical and relativistic definitions are equivalent although the interpretation of r may differ (see § 6.2). Further discussion to this issue will come later.

15. Verify that the solution found in exercise 13 satisfies the system given by Eqs. 139-141 (in the text). What this means?
    Answer: We have:

$$\begin{aligned}
e^{H} = e^{-K} &= 1 - \frac{C_s}{r} \\
H = -K &= \ln\left(1 - \frac{C_s}{r}\right) \\
H' = -K' &= \left(1 - \frac{C_s}{r}\right)^{-1} \frac{C_s}{r^2} \\
H'' &= -\left(1 - \frac{C_s}{r}\right)^{-2} \left(\frac{C_s}{r^2}\right)^2 - 2\left(1 - \frac{C_s}{r}\right)^{-1} \frac{C_s}{r^3}
\end{aligned}$$

Hence:

$$\begin{aligned}
\frac{H'K'}{4} - \frac{(H')^2}{4} - \frac{H''}{2} - \frac{H'}{r} &= -\frac{1}{4}\left(1 - \frac{C_s}{r}\right)^{-2} \left(\frac{C_s}{r^2}\right)^2 \\
&\quad -\frac{1}{4}\left(1 - \frac{C_s}{r}\right)^{-2} \left(\frac{C_s}{r^2}\right)^2 \\
&\quad +\frac{1}{2}\left(1 - \frac{C_s}{r}\right)^{-2} \left(\frac{C_s}{r^2}\right)^2 + \left(1 - \frac{C_s}{r}\right)^{-1} \frac{C_s}{r^3} \\
&\quad -\left(1 - \frac{C_s}{r}\right)^{-1} \frac{C_s}{r^3} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
-\frac{H'K'}{4} + \frac{(H')^2}{4} + \frac{H''}{2} - \frac{K'}{r} &= +\frac{1}{4}\left(1 - \frac{C_s}{r}\right)^{-2} \left(\frac{C_s}{r^2}\right)^2 \\
&\quad +\frac{1}{4}\left(1 - \frac{C_s}{r}\right)^{-2} \left(\frac{C_s}{r^2}\right)^2 \\
&\quad -\frac{1}{2}\left(1 - \frac{C_s}{r}\right)^{-2} \left(\frac{C_s}{r^2}\right)^2 - \left(1 - \frac{C_s}{r}\right)^{-1} \frac{C_s}{r^3} \\
&\quad +\left(1 - \frac{C_s}{r}\right)^{-1} \frac{C_s}{r^3} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
e^{-K}\left[1 + \frac{r\left(H' - K'\right)}{2}\right] - 1 &= e^{H}\left[1 + \frac{2rH'}{2}\right] - 1 \\
&= e^{H}\left[1 + rH'\right] - 1 \\
&= \left(1 - \frac{C_s}{r}\right)\left[1 + \left(1 - \frac{C_s}{r}\right)^{-1} \frac{C_s}{r}\right] - 1
\end{aligned}$$

$$\begin{aligned} &= 1-\frac{C_s}{r}+\frac{C_s}{r}-1 \\ &= 0 \end{aligned}$$

It means that the obtained Schwarzschild solution satisfies the vacuum Field Equations $R_{\mu\nu}=0$ (also see the next exercise).

16. Show that the definite form of the Schwarzschild metric that we obtained in the text (Eqs. 150-151) satisfies the vacuum Field Equations $R_{\mu\nu}=0$.
    Answer: In the text we obtained the system of Eqs. 139-141 as a result of employing the vacuum Field Equations in conjunction with the guessed Schwarzschild metric (Eqs. 122-123 in the text) which is given in terms of H and K and hence it is not fully determined. Moreover, in the previous exercise we verified that the system of Eqs. 139-141 is consistent provided that H and K are defined by the given expressions. We also justified in the text the given expression of C_s (i.e. $C_s=\frac{2GM}{c^2}$). So, we can conclude that the given definite form of the Schwarzschild metric (as given by Eqs. 150-151 in the text) satisfies the vacuum Field Equations (provided that H, K and C_s are defined properly by their given expressions).
17. Obtain the non-vanishing Christoffel symbols of the Schwarzschild metric in their definite form (refer to Eq. 126 in the text).
    Answer: Referring to the expressions of H,K,H',K' that we obtained in exercise 15 and noting that $C_s=\frac{2GM}{c^2}$, we have:

$$\begin{aligned}
\Gamma^0_{01}=\Gamma^0_{10}=\frac{1}{2}H' &= \frac{1}{2}\left(1-\frac{2GM}{c^2r}\right)^{-1}\frac{2GM}{c^2r^2}=\frac{GM}{c^2r^2}\left(1-\frac{2GM}{c^2r}\right)^{-1} \\
\Gamma^1_{00}=\frac{1}{2}e^{H-K}H' &= \frac{1}{2}\left(1-\frac{2GM}{c^2r}\right)^{2}\left(1-\frac{2GM}{c^2r}\right)^{-1}\frac{2GM}{c^2r^2}=\frac{GM}{c^2r^2}\left(1-\frac{2GM}{c^2r}\right) \\
\Gamma^1_{11}=\frac{1}{2}K'=-\frac{1}{2}H' &= -\frac{GM}{c^2r^2}\left(1-\frac{2GM}{c^2r}\right)^{-1} \\
\Gamma^1_{22}=-re^{-K}=-re^{H} &= -r\left(1-\frac{2GM}{c^2r}\right) \\
\Gamma^1_{33}=-re^{-K}\sin^2\theta &= -r\left(1-\frac{2GM}{c^2r}\right)\sin^2\theta \\
\Gamma^2_{12}=\Gamma^2_{21} &= \frac{1}{r} \\
\Gamma^2_{33} &= -\sin\theta\cos\theta \\
\Gamma^3_{13}=\Gamma^3_{31} &= \frac{1}{r} \\
\Gamma^3_{23}=\Gamma^3_{32} &= \cot\theta
\end{aligned}$$

18. Use the Schwarzschild metric to obtain a mathematical condition for the weak field approximation (i.e. where Newtonian gravity is a good approximation to general relativity).
    Answer: For the acceptability of the weak field approximation we should have an approximately flat spacetime (due to the weakness of gravity). This means that the factor $1-\frac{2GM}{c^2r}$ in the Schwarzschild metric should be close to unity (i.e. $1-\frac{2GM}{c^2r}\simeq 1$) so that the metric resembles the Lorentz metric of flat spacetime. Hence, the required condition is $\frac{2GM}{c^2r}\ll 1$ since in this case the Schwarzschild metric will reduce to the flat Lorentz metric with spatial spherical coordinates (see exercise 10 of §

2.5). We note that this condition applies for $r \geq R$ where R is the radius (or coordinate radius) of the gravitating object. It is obvious that the condition $\frac{2GM}{c^2 r} \ll 1$ can be written as $\frac{2GM}{c^2} \ll r$ which means that r is much larger than the Schwarzschild radius.

19. In § 1.3 we defined weak gravitational field as a field in which the gravitational potential energy of a test particle is negligible in comparison to its mass energy. Try to correlate this definition to the mathematical condition that is obtained in the previous exercise.
Answer: If we put the definition of § 1.3 in a mathematical form then we have:[123]

$$\frac{GMm}{r} \ll mc^2 \qquad \text{and hence} \qquad \frac{GM}{c^2 r} \ll 1$$

where M is the mass of the gravitating object and m is the mass of the gravitated object (i.e. test particle). As we see, this condition is identical to the condition of the previous exercise apart from a factor of 2 which is negligible at this level of approximation.

20. What is the essence of the Birkhoff theorem?
Answer: The essence of the Birkhoff theorem is that the Schwarzschild solution is the only solution to the vacuum Field Equation subject to the boundary condition at infinity (i.e. asymptotic flatness) and other stated conditions.[124] In fact, this applies even if the gravitating object varies in time as long as the variation is spherically symmetric. In other words, the solution in the vacuum region will not be affected by dynamic variations (e.g. expansion or contraction) in the gravitating object as long as these variations are spherically symmetric. So, as long as the matter distribution of the gravitating body is spherically symmetric the geometry of spacetime in the vacuum region outside the gravitating body is described by the Schwarzschild metric (which is time independent) even if the matter distribution is time dependent (i.e. it varies in time). From this perspective, Schwarzschild solution resembles Newtonian gravity.

4.1.2 Geodesic Equation in Schwarzschild Metric

1. Summarize the essence of the formalism and strategy of general relativity in determining the trajectories of massive and massless objects in the spacetime under the effect of gravity. Hence, justify the investigation of this subsection and its applications.
Answer: In general relativity, the problem of finding the spacetime trajectories of physical objects under the effect of gravity is reduced to the problem of finding the geodesic paths in a 4D spacetime where the metric of the spacetime is determined from the physical setting. Accordingly, the investigation of this subsection is justified by the need to develop the geodesic equations in the Schwarzschild spacetime which is the appropriate spacetime for describing the motion of massive and massless objects in many practical situations, e.g. the planetary motion and the deflection of light by the Sun (see § 5.2 and § 8.2).

2. Compare the system of Eqs. 161-163 (in the text) to the system of Eqs. 164-166 (in the text).
Answer: We note two main differences:
(**a**) The term $\frac{GM}{B^2}$ in Eq. 162 is absent in Eq. 165.
(**b**) In Eqs. 161-163 the affine parameter is τ (which essentially represents line element since $\tau = \sigma/c$), while in Eqs. 164-166 it is λ (which does not represent line element since the geodesic in this case is null and hence $\sigma = 0$). In fact, this difference should have an impact on the physical meaning of the symbols A and B in the two systems (as can be seen from their physical dimensions). So, although (for simplicity) we use the same symbols in the two systems they should be interpreted differently.

4.2 Kerr Solution

1. Summarize the main characteristics of the Kerr solution.
Answer: We note the following:

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[123] In fact, we are taking the magnitude of the potential energy (since it is negative).

[124] The Birkhoff theorem may be stated (more generally) as: the Schwarzschild solution is the only spherically symmetric solution to the vacuum Field Equation.

• It is an extension to the Schwarzschild solution where the gravitating object in the Schwarzschild problem is assumed to be rotating around its axis of symmetry. Hence, it converges to the Schwarzschild solution when the gravitating object is non-rotating (assuming that the gravitating object is spherically symmetric rather than axially symmetric).[125]
• It is a vacuum solution of the Field Equation.
• It is time independent solution, i.e. it is steady state solution although the gravitating object is rotating.
• Unlike the Schwarzschild solution, the gravitating object in the Kerr solution is required to have axial (rather than spherical) symmetry (noting that spherical symmetry is a special case of axial symmetry and hence it satisfies axial symmetry).[126]
• The Kerr solution is axially symmetric (unlike the Schwarzschild solution which is spherically symmetric).
• Like the Schwarzschild solution, it converges to the Lorentz metric of special relativity at infinity, i.e. $r \to \infty$.
• Like the Schwarzschild solution, it converges to the Lorentz metric of special relativity when mass vanishes, i.e. $M = 0$.
• Any solution of the Kerr problem is equivalent to the Kerr solution, i.e. the solution is unique.
• The gravitating object in the Kerr solution is characterized physically by mass M and angular momentum J.
• The Kerr metric tensor is not diagonal and hence its line element contains mixed differential terms, i.e. $cdt\,d\phi$.
• It predicts frame dragging (see § 8.9).
• It is used in the study of rotating black holes (see § 8.7 and § 8.7.2).
• The Kerr solution implies the existence of two event horizons (inner and outer) as well as an ergo-region (or ergo-sphere) in the space that surrounds a rotating black hole (see § 8.7 and § 8.7.2).

2. Write down the Kerr metric tensor.
Answer:

$$[g_{\mu\nu}] = \begin{bmatrix} 1-\frac{R_S r}{\rho^2} & 0 & 0 & \frac{R_S r a \sin^2\theta}{\rho^2} \\ 0 & -\frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & -\rho^2 & 0 \\ \frac{R_S r a \sin^2\theta}{\rho^2} & 0 & 0 & -\sin^2\theta\left(r^2+a^2+\frac{R_S r a^2 \sin^2\theta}{\rho^2}\right) \end{bmatrix}$$

3. Show that the Kerr solution converges to the Schwarzschild solution when the gravitating object is not rotating.
Answer: When the gravitating object is not rotating its angular momentum is zero (i.e. $J = 0$) and hence we have:

$$\begin{aligned} a &= \frac{J}{cM} = 0 \\ \rho^2 &= a^2\cos^2\theta + r^2 = r^2 \\ \Delta &= r^2 - rR_S + a^2 = r^2 - rR_S \end{aligned}$$

On substituting from these equations into the Kerr quadratic form we get:

$$\begin{aligned} (d\sigma)^2 &= \left(1-\frac{R_S r}{r^2}\right)(cdt)^2 - \frac{r^2}{r^2-rR_S}(dr)^2 - r^2(d\theta)^2 - \sin^2\theta\left(r^2+0+\frac{0}{r^2}\right)(d\phi)^2 + \frac{0}{r^2}cdt\,d\phi \\ &= \left(1-\frac{R_S}{r}\right)(cdt)^2 - \left(1-\frac{R_S}{r}\right)^{-1}(dr)^2 - r^2(d\theta)^2 - r^2\sin^2\theta\,(d\phi)^2 \end{aligned}$$

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[125] As indicated earlier, Schwarzschild geometry may be generated in some cases by non spherically symmetric matter distribution. However, these are exceptional cases and hence they should not affect the default requirement.

[126] In axial symmetry the metric is independent of ϕ while in spherical symmetry the metric is independent of ϕ and θ and hence spherical symmetry is a special case of axial symmetry where the metric is independent of θ as well.

which is the Schwarzschild quadratic form (see § 4.1.1).
Note: we should assume that the gravitating object is spherically symmetric rather than axially symmetric (because Schwarzschild requires spherical symmetry). We should also assume that the Boyer-Lindquist coordinates reduce to the Schwarzschild coordinates in this case (which is the case since $a = 0$; see exercise 10).

4. Show that the Kerr metric converges to the Lorentz metric of special relativity when $M = 0$.
Answer: We can achieve the condition $M = 0$ in two steps:
(**a**) We stop the rotation of the Kerr object first and hence we obtain a Schwarzschild metric according to the previous exercise.[127]
(**b**) We then reduce the mass until it vanishes (i.e. $M = 0$) and hence we obtain the Lorentz metric of special relativity according to exercise 7 of § 4.1.1 where it was shown there that the Schwarzschild metric converges to the Lorentz metric (with spatial spherical coordinates) when $M = 0$.
Note: achieving the condition $M = 0$ in two steps may look more sensible if we write $a = \frac{J}{cM}$ as $J = acM$. In brief, there is no angular momentum when $M = 0$ (and hence there is no rotation), and so instead of doing this in a single step where both M and J vanish simultaneously, we do it in two steps where J vanishes first (by vanishing the *rotation parameter* a) and M vanishes second.
5. Show that the Kerr metric is time independent.
Answer: Referring to exercise 2, all the coefficients of the Kerr metric tensor are time independent (i.e. there is no t in any coefficient) and hence the Kerr metric is time independent.
6. Show that the Kerr metric is axially symmetric but not spherically symmetric.
Answer: To have axial symmetry the metric should be independent of ϕ while to have spherical symmetry the metric should be independent of ϕ and θ.[128] Referring again to exercise 2, the coefficients of the metric tensor are independent of ϕ but not independent of θ and hence it is axially (but not spherically) symmetric. This should be physically sensible since the gravitating object is rotating and hence there is a preferential direction (along the axis of rotation) for the θ coordinate (which we may imagine to be somewhat similar to the usual θ in spherical coordinates) but not for the ϕ coordinate. This is because θ is related to the orientation relative to the axis while ϕ is related to the rotation around the axis.
Note: in problem 2 of § 4.1.1 we claimed that the presence of θ in the Schwarzschild metric does not affect the spherical symmetry, while in the present exercise we claimed that the presence of θ in the Kerr metric does affect the spherical symmetry. So, we need to justify the difference.
In fact, the difference arises from the physical information that the axis by which θ is defined in the Schwarzschild metric is arbitrary since it can be oriented in any direction without affecting the physical setting, while the axis by which θ is defined in the Kerr metric is not arbitrary since it is the axis of rotation and hence changing the orientation of this axis will introduce a real change to the physical setting of the problem and solution. In other words, the choice of the axis and θ in the Schwarzschild metric is a matter of coordinates and hence it is of purely conventional value, while the choice of the axis and θ in the Kerr metric is a matter of physics and hence it has real physical value. This highlights the issue that in physical sciences the mathematical formalism should be read and interpreted sensibly and in the light of the available physical information.
7. Investigate the singularities of the Kerr solution.
Answer: As we can see from the expression of $(d\sigma)^2$ of the Kerr solution, we have one singularity when $\rho = 0$ and another singularity when $\Delta = 0$.[129] According to the literature, the singularity at $\rho = 0$ is essential while the singularity at $\Delta = 0$ is accidental. In fact, there are many details about the nature of these singularities and what they represent in spacetime. However, this is of little interest to us and hence we refer the readers to the literature of Kerr solution for details.

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[127] In fact, this should require the assumption of spherical symmetry which may affect the generality of the result.

[128] In fact, we put it in this way for simplicity. However, in Boyer-Lindquist coordinates even the radial variable violates spherical symmetry (see exercise 10).

[129] This is based on pure mathematical considerations. Whether this is physically applicable and sensible (i.e. these conditions correspond to vacuum solution and hence they are within its domain) or not should depend on further physical considerations and verifications. For example, the situation in ordinary objects could differ from the situation in black holes.

8. Is the Kerr line element reversible in time and why?
   Answer: No, because there is a $cdt\, d\phi$ mixed term in the Kerr line element and hence it is not invariant under the transformation $dt \leftrightarrow -dt$. This should be physically sensible because rotation has a specific sense (say clockwise or anticlockwise) and hence it should not be invariant under such time reversal. Irreversibility also applies to the azimuthal variable ϕ due to rotation.
9. Why is there a $cdt\, d\phi$ mixed term in the Kerr line element?
   Answer: From a formal perspective, this is because the Kerr metric tensor is not diagonal. From a physical perspective, this reflects the aforementioned fact in the previous exercise about the irreversibility in time and ϕ (which originates from having rotation and angular momentum).
10. What are the Boyer-Lindquist coordinates?
    Answer: The Boyer-Lindquist coordinates are defined as follows: ct and ϕ have their ordinary meaning (i.e. c is the speed of light, t is the time coordinate and ϕ is the familiar azimuthal coordinate as in the spherical coordinates) while r and θ are defined in terms of the corresponding Cartesian coordinates x, y, z by the following relations:

$$\begin{aligned} x &= \sqrt{a^2 + r^2} \sin\theta \cos\phi \\ y &= \sqrt{a^2 + r^2} \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned}$$

    We note that the surfaces of constant t and constant r in the Boyer-Lindquist coordinates are not spheres (as can be easily checked from the Kerr quadratic form).
    Note: the above interpretation of the Boyer-Lindquist coordinates in the Kerr metric holds only in the limit of vanishing gravitating mass (which is equivalent to $R_S \to 0$) where the spacetime tends to the Minkowskian geometry. The reader should note that this is consistent with what is in exercise 4 taking into account that in exercise 4 we followed a two-step approach.
11. What is the difference between "static" and "stationary" (as part of the terminology used in the discussion of space metric)?
    Answer: There is no universal convention about the meaning of these terms. However, in the literature of general relativity (within the context of discussing spacetime metrics) "static" is commonly used to mean that the metric tensor is time independent and the quadratic form is invariant under time reversal (i.e. $dt \leftrightarrow -dt$), while "stationary" is commonly used to mean that the metric tensor is time independent but the quadratic form is not (or may not be) invariant under time reversal. Accordingly, Schwarzschild solution is "static" while Kerr solution is "stationary". However, the reader should be vigilant to identify the convention of each individual author. By the way, in this book we do not follow the aforementioned terminology although we may use it occasionally with notification.
    Note: the aforementioned terminology may be adapted (following some suggestions in the literature) and hence one might say: Schwarzschild metric tensor is stationary and its quadratic form is static, while Kerr metric tensor is stationary and its quadratic form is also stationary (i.e. it is not static since it is not reversible in time due to rotation).

4.3 Reissner-Nordstrom Solution

1. Summarize the main characteristics of the Reissner-Nordstrom solution.
   Answer: We note the following:
   - It is an extension to the Schwarzschild solution where the gravitating object in the Schwarzschild problem is assumed electrically charged. Hence, it converges to the Schwarzschild solution when the object becomes electrically neutral.
   - It is not a vacuum solution.
   - Historically, it is the second solution to the Field Equations.
   - It is spherically symmetric and time independent solution.
   - It converges to the Lorentz metric of special relativity at infinity, i.e. $r \to \infty$.
   - Any solution of the Reissner-Nordstrom problem is equivalent to the Reissner-Nordstrom solution

(i.e. the solution is unique).
• The gravitating object in the Reissner-Nordstrom solution is characterized physically by mass M and electric charge Q.
• It may be used in the speculations about black holes.

4.4 Kerr-Newman Solution

1. Summarize the main characteristics of the Kerr-Newman solution.
   Answer: We note the following:
   • It is an extension to the Schwarzschild solution where the gravitating object in the Schwarzschild problem is assumed rotating around its axis of symmetry and electrically charged. Hence, it converges to the Schwarzschild solution when the object is non-rotating and electrically uncharged (assuming it is spherically symmetric), to the Kerr solution when the object is electrically uncharged, and to the Reissner-Nordstrom solution when the object is non-rotating (assuming it is spherically symmetric). In other words, Schwarzschild, Kerr and Reissner-Nordstrom solutions are special cases of Kerr-Newman solution.
   • It is not a vacuum solution.
   • Like the Kerr solution, it predicts frame dragging (see § 8.9).
   • The gravitating object in the Kerr-Newman solution is characterized physically by mass M, angular momentum J and electric charge Q.
   • Unlike the Schwarzschild and Reissner-Nordstrom solutions, the Kerr-Newman solution has axial (rather than spherical) symmetry and hence it is like the Kerr solution in this regard.
   • It converges to the Lorentz metric of special relativity at infinity, i.e. $r \to \infty$.
   • The Kerr-Newman metric tensor is not diagonal (and hence its line element contains mixed differential terms).
   • It may be used in the speculations about black holes.
2. Compare the Schwarzschild metric to the following three metrics: Kerr, Reissner-Nordstrom, and Kerr-Newman.
   Answer: All these three metrics are extensions to the Schwarzschild metric and hence the physical setting of the gravitating object and the surrounding spacetime is similar to the physical setting in the Schwarzschild problem. However, the gravitating object in the Kerr metric is distinguished by being rotating around its axis of symmetry (unlike Schwarzschild object which is non-rotating), the gravitating object in the Reissner-Nordstrom metric is distinguished by being electrically charged (unlike Schwarzschild object which is electrically uncharged), and the gravitating object in the Kerr-Newman metric is distinguished by being rotating around its axis of symmetry and electrically charged (unlike Schwarzschild object which is non-rotating and electrically uncharged). We should note that the gravitating object in the Kerr and Kerr-Newman problems require axial symmetry (rather than the more restrictive spherical symmetry of Schwarzschild problem) since the object is rotating.
3. Referring to black holes, what is the main physical distinction of the Kerr and Kerr-Newman metrics from the Schwarzschild metric?
   Answer: The main physical distinction of these metrics is the modification of the event horizon of black holes that are described by these metrics where these metrics introduce changes to the number and shape of event horizon of the Schwarzschild black hole as a result of rotation and angular momentum (see § 8.7 and § 8.7.2). For the same reason, these metrics predict frame dragging effect (although this effect is not restricted to black holes). These metrics also introduce an ergo-region which does not exist in the Schwarzschild metric. We should also note that the singularity in these metrics becomes a ring rather than a point as in the case of the Schwarzschild metric (see § 8.7.2).

Chapter 5
Classical Limit of General Relativity

5.1 Convergence to Newtonian Gravity

1. What are the main factors that determine the intensity (or strength) of the gravitational field?
   Answer: Strong gravitational fields are generated by very massive or very compact gravitating objects like black holes. However, the intensity of the gravitational field also depends on the distance from the source of gravity due to the inverse square relation (classically at least). So, the main factors are the mass of the gravitating object and its compactness as well as the distance from it. Accordingly, we should find strong gravitational fields in the close vicinity of very massive or compact objects. In fact, this can be concluded from Newton's gravity law (i.e. direct relation with mass and inverse relation with distance squared) or Poisson's gravity equation (or rather more obviously from its equivalent equation $\nabla \cdot \mathbf{g} = -4\pi G\rho$ possibly with the aid of the divergence theorem).
2. Considering that classical gravity is basically a spatial phenomenon with no time dependency, can we have time dependency in classical gravity? If so, what is the difference between time dependency in classical gravity and in general relativity?
   Answer: Yes, of course. In fact, we have two main types of time dependency in classical gravity:
   (**a**) Time dependency in the gravitational force but not in the gravitational field. For example, we may have a gravitated object approaching or receding from a static gravitating object and hence the gravitational force is time dependent. However, the gravitational field in this case is time independent.
   (**b**) Time dependency in the gravitational field itself. For example, we may have a gravitating object that contracts or expands in a spherically asymmetric way or a gravitating object whose mass increases or decreases or a number of gravitating objects that interact with each other and hence the gravitational field changes with time in these cases.
   So, the difference between time dependency in classical gravity and in general relativity is about the basic formulation of the gravitational phenomenon where the classical formulation (as represented by Newton's law or Poisson's equation) is inherently and purely spatial (although time dependency can be introduced in its applications and instantiations accidentally), while the general relativistic formulation is fundamentally tempo-spatial and hence time is a fundamental component of the gravitational phenomena (although time independence may be introduced in its applications and instantiations accidentally in special circumstances). In fact, this originates from the difference in the source of gravity in the two theories where the source of gravity in classical physics is mass which is essentially spatial, while the source of gravity in general relativity is energy-mass which is tempo-spatial since energy corresponds to time (noting that the temporal component of the momentum 4-vector is E/c) as explained earlier (see § 1.1 and § 2.14; also see B4).
3. The condition of time independence may be stated by some as slow variation in time. Comment on this.
   Answer: This is because slow variation in time means that the gravitational field has weak dependency on time and hence it is virtually time independent over the time intervals of interest. In other words, over a sufficiently small time intervals (which are of interest to the observer) the variation is too tiny to be of any significance (considering the observational limits on accuracy) and hence it can be ignored. However, our expression (i.e. "time independence" instead of "slow variation") should be sufficiently general to include this case if we consider slow variation as an approximation to time independence.
4. Does the condition of slow speed relative to c contradict the time independence of the gravitational field?
   Answer: No, because time independence belongs to the gravitating object (i.e. to the source of gravity and hence to the gravitational field) while variation in time (which may be implied by "slow speed" or "slow motion") belongs to the gravitated object.

5. Outline, roughly and in general terms, the situations where general relativity can be replaced by classical gravity.
Answer: In brief, classical gravity is a valid approximation to general relativity when the spacetime is essentially flat and the variation over space or/and time is minor. Accordingly, we expect classical gravity to replace general relativity in the case of weak slowly-varying (or static) fields with no involvement of large distances or large time intervals or high speeds. In other words, we expect the failure of classical gravity to replace general relativity in the cases of strong fields or fast temporal or spatial variations or involvement of large distances or large time intervals or involvement of high speeds (e.g. light). In fact, most of these cases may be seen as instances or demonstrations of the violation to the locality condition (in the extensive sense of locality) by having extended space or/and time or considerable temporal or spatial variations where the effects of non-locality become evident.
6. Use the approximation $g_{00} \simeq \eta_{00} + \varepsilon_{00} = 1 - \frac{2GM}{c^2 r}$ which is given in the text to propose a formal condition for the validity of classical gravity. Also interpret this condition.
Answer: We can start from the premise that the classical theory of gravity applies when the gravitational field is weak and this means that the spacetime is approximately flat. Formally, being approximately flat means $g_{00} \simeq 1$ and this implies $\frac{2GM}{c^2 r} \simeq 0$.[130]
We may interpret this condition as $\frac{2GM}{c^2 r} \equiv \frac{R_S}{r} \simeq 0$ (where R_S is the Schwarzschild radius of the gravitating object). In other words, classical gravity is valid when $r \gg R_S$. We may also interpret it in other ways (refer for example to § 1.3 and exercise 19 of § 4.1.1; also see exercise 18 of § 4.1.1). However, all these interpretations are generally similar or equivalent and they usually depend on the circumstances and contexts.
7. Justify the claim that in the classical limit we have $d\tau \simeq dt$ considering both the special and general relativistic effects.
Answer: The classical approximation $d\tau \simeq dt$ considers both the kinematical time dilation of special relativity (since in the classical limit $\gamma \simeq 1$ due to the slow-speed requirement) and the gravitational time dilation of general relativity (since in the classical limit the metric of general relativity converges to the Minkowski metric of special relativity where no gravitational time dilation exists). Accordingly, in the classical limit we have $d\tau \simeq dt$, as claimed.
8. Obtain (using a compact method) the classical equation of gravitational field from the geodesic equation of general relativity in the classical limit.
Answer: We start from the geodesic equation, that is:

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0$$

In the classical limit we have $\tau = t$ and all the terms of the sum $\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}$ are negligible in comparison to $\Gamma^\alpha_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau}$, moreover classically we do not have a temporal component, i.e. $\alpha \equiv i = 1, 2, 3$. Hence, the geodesic equation becomes:

$$\begin{aligned}
\frac{d^2x^i}{dt^2} + \Gamma^i_{00}\frac{dx^0}{dt}\frac{dx^0}{dt} &= 0 \\
\frac{d^2x^i}{dt^2} + c^2\Gamma^i_{00}\frac{dt}{dt}\frac{dt}{dt} &= 0 \\
\frac{d^2x^i}{dt^2} + c^2\Gamma^i_{00} &= 0 \\
\frac{d^2x^i}{dt^2} + c^2\left(-\frac{1}{2}\partial^i\varepsilon_{00}\right) &= 0 \\
\frac{d^2x^i}{dt^2} &= \frac{c^2}{2}\partial^i\varepsilon_{00}
\end{aligned}$$

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[130] We are using only the g_{00} component in the formulation of this condition. This should be justified by being in the classical limit which relies only on this component (as discussed in the text). We should also note that the above condition (i.e. $\frac{2GM}{c^2 r} \simeq 0$) can also be obtained more directly from the condition $|\varepsilon_{\mu\nu}| \ll 1$ which we stated earlier in the text.

$$\begin{aligned}\frac{d^2x^i}{dt^2} &= \partial^i\left(\frac{c^2}{2}\varepsilon_{00}\right)\\ \frac{d^2x^i}{dt^2} &= \left[-\nabla\left(\frac{c^2}{2}\varepsilon_{00}\right)\right]^i\\ \frac{d^2x^i}{dt^2} &= -\left[\nabla\left(\frac{c^2}{2}\varepsilon_{00}\right)\right]^i\end{aligned}$$

where in line 2 we use $x^0 = ct$, in line 4 we use the approximation $\Gamma^i_{00} \simeq -\frac{1}{2}\partial^i\varepsilon_{00}$ which we obtained in the text (with $\alpha \equiv i$), in line 6 we use the constancy of $\frac{c^2}{2}$, and in line 7 we use the bracket notation to express the i^{th} component of gradient in contravariant form (using contravariant nabla).[131] Now, if $\frac{c^2}{2}\varepsilon_{00}$ represents the gravitational potential Φ then the last line becomes:

$$\frac{d^2x^i}{dt^2} = -\left[\nabla\Phi\right]^i$$

which is the classical equation of the gravitational field noting that this equation is equivalent to:

$$\mathbf{a} = -\nabla\Phi = \mathbf{g}$$

which is the essence of the (classical) equivalence principle.

5.2 Planetary Motion

1. Give some examples of necessary assumptions that should be made about the gravitating object for the validity of employing the classical limit in the planetary motion.
   Answer: For example:
   • We should assume that the gravitating object essentially meets the conditions of Schwarzschild object (such as staticity and spherical symmetry as given in § 4.1) because we are using the Schwarzschild-based geodesic equation to obtain the general relativistic formulation in the classical limit (although this is not specific to the classical limit).
   • We should also assume that the gravitating object essentially consists of matter (since energy is not a classical source of gravity).
   Both these assumptions are generally valid (at least approximately) in most classical orbiting systems (and in the solar system in particular).
2. Give some examples of necessary assumptions that should be made about the gravitated object for the validity of employing the classical limit in the planetary motion.
   Answer: For example:
   • We should assume that the gravitated object has negligible mass in comparison to the mass of the gravitating object so that the spacetime geometry which is created by the gravitating object is not affected (i.e. the presumed Schwarzschild geometry created by the gravitating object is not distorted). In fact, this is very good assumption in the case of planetary motion in the solar system (and indeed in many astronomical orbiting systems) since the planets have negligible mass in comparison to the Sun. For example, the mass of the most massive planet, which is Jupiter, is about 1.898×10^{27} kg while the mass of the Sun is about 1.989×10^{30} kg.
   • We should also assume that the speed of the gravitated object is very small compared to c (to exclude non-classic special and general relativistic effects).[132] This is also very good assumption in the case of planetary motion in the solar system (as well as many astronomical orbiting systems). For example, the orbital speed of the fastest planet, which is Mercury, is about 48 km/s which is negligible in comparison to $c \simeq 3 \times 10^5$ km/s.

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[131] In fact, we are using the spatial part of the contravariant nabla 4-operator (see exercise 15 of § 2.2).

[132] As seen in note 3 of problem 1, this condition is required for discarding the term $\frac{3GMu^2}{c^2}$ (and hence for the validity of the formulation of the classical limit).

Chapter 6
Frames, Coordinates and Spacetime

1. Discuss the consequences of the dependency of the results that we obtain in this chapter on the Schwarzschild metric.
   Answer: This dependency should have an impact not only on the theoretical validity of these results (i.e. they may not be valid theoretically in other metric spaces) but it should have observational and practical consequences. For example:
   • The experimental evidence may support the results of a certain metric in a given physical setting but not the results of another metric and hence these differences can be used in verifying the theory and inspecting its consequences in different physical situations.
   • If the experimental results endorse the Schwarzschild metric in a given situation then we can conclude that this metric is the right metric for the particular physical setting.
   • If the results in certain physical situations (whether similar or different) lead to contradiction about the appropriate metric (e.g. in a given physical setting some results are consistent with a certain metric while others contradict this metric and agree with other metric) then we may infer an inconsistency or a defect in the theory itself.
   • An inconsistency or a defect in the theory may also be inferred if the observational results of a given physical system contradict the general relativistic predictions that are based on the relevant metric of that system according to the theory.

6.1 Frames in General Relativity

1. The frames in classical mechanics and in special relativity are usually global while the frames in general relativity are generally local. Why?
   Answer: The spacetime in classical mechanics[133] and in special relativity is flat and uniform and hence a single global frame that covers the entire spacetime is sufficient. But in general relativity the spacetime is curved and non-uniform; moreover it is required to apply special relativity at local level (at least in some frames). Therefore, multiple local frames are needed to implement the plan of general relativity.
2. Is it necessary that the frames in general relativity are local? Why?
   Answer: Yes, it is necessary but out of choice and for practical reasons rather than out of inherent necessity. In other words, we can adopt a global frame in general relativity but it will not be useful for implementing the plan of general relativity which is based on applying special relativity locally. More clearly, general relativity is based on local application of special relativity and hence we need only local frames that coordinate locally-flat patches of spacetime to implement the plan of general relativity.
   Note: as we will see later, the coordinate variables can be seen as a basis for a global frame in contrast to the physical (or metrical) variables which are the basis for the local frames.
3. If there are frames in general relativity that are not subject to the rules of special relativity (i.e. gravitational frames that are not in a state of gravitational free fall) then how are these frames treated in general relativity? Can this be a source of defect or limitation in the theory of general relativity?
   Answer: As we saw in the Problems, the position of general relativity (or at least general relativists) with regard to this type of frames is not very clear (although it seems to be addressed by the *Principle of General Covariance* whose validity and logical sensibility are questionable). It seems to us that this can be a source of defect or limitation in the theory of general relativity. If so, then we may still need to assume the existence of gravitational forces which are allegedly made redundant by the paradigm of geodesic. Also, see § 10.1.11.

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[133] In classical mechanics "spacetime" should mean space and time. In fact, we are using "spacetime" for correspondence and comparison with the relativity theories.

4. What are the cases in which special relativity applies in the spacetime of general relativity?
   Answer: There are two main cases:
   (**a**) At infinity (i.e. very far from any source of gravitation) due to the absence of gravity and hence the spacetime is actually flat.
   (**b**) Locally anywhere in the spacetime due to the local flatness of the spacetime. However, the application in this case requires free fall to annul the effect of gravity. We note that the locality condition means that the region of spacetime in which special relativity applies should be sufficiently small so that the field gradient and tidal forces become negligible (and hence the free fall can annul the effect of "uniform" gravity because free fall cannot annul the field gradient and tidal forces) and this depends on the form and intensity of the field gradient. In other words, the locality ensures the "uniformity" of the gravitational field[134] while the free fall ensures the annulment of the "uniform" gravity by the free fall acceleration (thanks to the equivalence principle). In fact, this should endorse the assertion that special relativity cannot apply even locally in non freely falling frames in the neighborhood of a source of gravity because in such frames although the gravity is "uniform" (due to the locality) there is no free fall acceleration to annul the effect of the "uniform" gravity. In other words, in the absence of free fall the frame is not effectively inertial even though the spacetime is (locally) flat and hence special relativity (which is based on both the flatness of spacetime and the inertiality of frame) will not apply.
5. For the application of special relativity, what conditions we need to assume about the spacetime and the reference frame? Try to link this to the application of special relativity locally in freely falling frames in the spacetime of general relativity.
   Answer: We need two main conditions: the spacetime is flat and the frame is inertial. The flatness condition is required because special relativity does not deal with the effects of gravity (which, according to the principle of metric gravity, distorts the spacetime), while the inertiality condition is required because special relativity does not deal with the effects of acceleration. Accordingly, if the spacetime is curved or/and the reference frame is not inertial then we need general relativity.
   Regarding the application of special relativity in the spacetime of general relativity, the locality is supposed to ensure flatness (since Riemannian space, which the spacetime of general relativity is supposed to be, is locally flat) while free fall is supposed to ensure inertiality (since the acceleration of free fall cancels the effect of gravity according to the equivalence principle).[135]
6. Discuss the status of accelerating frame at infinity (i.e. very far away from any source of gravity) in the spacetime of general relativity.
   Answer: From the answer of the previous exercise we can say: although the spacetime at infinity is flat, the rules of special relativity do not apply in such a frame because of the restriction of special relativity to inertial frames. In fact, we can add this sort of frames as a fourth type of local frames in the spacetime of general relativity (noting that this type of frames is not of much interest to general relativity as a gravity theory and hence we ignored it in our classification of the main types of local frame). Also see the next exercise.
7. In the text we considered three main types of local frame in the spacetime of general relativity. Try to elaborate on this.
   Answer: In fact, we can consider local frames in more detail and hence we may produce the following list of local frames (noting that even this list can be elaborated further):
   • Non-accelerating frame at infinity (regardless of being stationary or moving uniformly and relative to what). Special relativity does apply in this frame.
   • Accelerating frame at infinity (regardless of what "accelerating" means and if it is relative to absolute frame or to the source of gravity). Special relativity does not apply in this frame.

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[134] To be closer to the terminology and spirit of general relativity we should say: the locality ensures the "flatness" of the spacetime (noting that according to the principle of metric gravity the "gravitational field" is represented by the curvature of spacetime and hence the uniformity of the gravitational field resembles the flatness of the spacetime).

[135] To be more clear and consistent we should say: since the acceleration of free fall cancels the effect of the acceleration that is equivalent to gravity according to the equivalence principle (and thus the cancellation of the effect of the acceleration that is equivalent to gravity ensures inertiality).

- Freely falling frame in gravitational field. Special relativity does apply in this frame.
- Stationary frame (relative to the source of gravity) in gravitational field. Special relativity does not apply in this frame.
- Moving frame (relative to the source of gravity) in gravitational field but it is not freely falling. Special relativity does not apply in this frame.

Note: when we talk about non-applicability of special relativity we mean as it is, although it may apply through transformation from a frame in which special relativity apply (at least in some cases).[136] Accordingly, the transformed form of any law may be seen as a modified special relativistic law (where the formal transformation supposedly transforms the actual physical effect that distinguishes the two frames).

6.2 Coordinates in General Relativity

1. What distinguishes general relativity from classical mechanics and special relativity and hence makes the coordinates in general relativity of symbolic rather than metrical or physical value?
   Answer: The spacetime in classical mechanics and in special relativity is flat and hence it is sufficiently simple to be described by a single set of variables, i.e. the coordinate variables and their physical counterparts are identical (e.g. the coordinate time is the same as the physical time) or at the worst they can be correlated by simple linear relations. However, the situation in general relativity is more complicated because the spacetime of general relativity is curved and complex and hence two sets of variables (i.e. coordinate and physical) are needed.
2. Can we fix the problems that potentially arise from the premise that the coordinates of spacetime in general relativity have no metrical or physical significance?
   Answer: If we take the Schwarzschild spacetime as an example then we will see later that the radial coordinate r and the temporal coordinate t converge to the physical radial length and time at infinity (refer to § 6.3.2 and § 6.4.2) while θ and ϕ are identical to their counterparts in spherical coordinates (refer to § 6.4.1). Therefore, we may claim that the coordinates of spacetime (at least in the Schwarzschild metric) have physical significance (i.e. they represent the physical quantities at infinity) although the local time and length should be obtained through the metrical relations with the use of the coordinates of spacetime.
   However, this fix may establish a relation between the physical variables at infinity and the coordinate variables but it does not create physical significance to the coordinate variables on their own. In other words, these variables still represent non-physical quantities at every location in spacetime except at infinity. Anyway, this fix does not address the practical problems (as explained in the text) that originate from having two sets of variables (i.e. coordinate and physical).
3. Assess the coordinates of rectilinear and curvilinear systems of 3D flat Euclidean space, e.g. Cartesian, cylindrical and spherical coordinate systems.
   Answer: These coordinates (regardless of being rectilinear or curvilinear) have metrical significance because they are ordinary coordinates of a flat space. In other words, they do not represent a curved space as in general relativity where the coordinates lose their metrical value according to the theory and its implications. In brief, being rectilinear or curvilinear is irrelevant from this perspective as long as the coordinated space is flat.
4. Use a specific metrical relation as an example to elucidate the issue that in real life we have access only to physical (rather than coordinate) variables.
   Answer: Suppose that we want to use the metrical relation $d\tau = \left(1 - \frac{2GM}{c^2 r}\right)^{1/2} dt$ to obtain the physical quantity $d\tau$. To do this we need to have r and dt which are coordinate variables related to specific events. However, observed events in real life are identified by physical (rather than coordinate) quantities. This means that the r and dt that are needed in the above metrical relation to obtain $d\tau$ may not be available.

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[136] We note that this may no apply in some cases such as the case of accelerating frame at infinity, and hence this may be regarded as a theoretical gap in general relativity as a "General Theory".

5. Address the problem that modeling gravity as a physical phenomenon in a flat spacetime (rather than a geometric attribute of a curved spacetime) may be impossible or non-physical if the actual physical spacetime is not flat.
   Answer: We note the following:
   • If the spacetime is curved then this is not specific to gravity but it should include all physical theories and phenomena that take place in spacetime. In other words, regardless of the nature of the spacetime (whether flat or curved) there is no reason for singling the gravity out and treating it differently as a curvature of spacetime.
   • There is no substantial evidence for the curvature of spacetime. At least within the dimensions of the known Universe that we live in, the spacetime is effectively flat and hence this should be considered as a physical fact. The development of science so far (apart from general relativity and its applications such as relativistic cosmology) is largely based on the assumption of flat spacetime, and this can be seen evidently in all major physical theories (whether classical or modern) such as classical mechanics, Lorentz mechanics and quantum mechanics which employ flat spacetime in their models and formulations. In fact, the potentially claimed curvature of spacetime of our Universe is a product of the application of general relativity in cosmology and hence it is not a fact proven independently of general relativity.
   • Even if we accept the reality of spacetime, we should not accept the reality of coordinate systems and frames of reference and hence the physical spacetime can be coordinated by "flat" coordinate systems and frames (even if it is presumed curved) where any supposed curvature or non-linearity of spacetime can be formulated and interpreted as a physical effect in flat spacetime rather than a geometric attribute of curved spacetime.[137] In brief, as physicists we are not obliged to follow a strictly geometric methodology and approach.
   • To sum up, the curvature of spacetime is a mere geometric speculation and hypothesis and hence it is foreign to physics. In other words, physics does not need this geometric paradigm and methodology to develop realistic and accurate physical models that describe the Universe. Yes, the curvature of spacetime may be a demand and need for some physicists who approach science from their own personal perspectives and preferences.

6.3 Time in Schwarzschild Spacetime

6.3.1 Relation between Coordinate Time and Proper Time

1. What the condition $\frac{2GM}{c^2 r} < 1$ means?
   Answer: The condition $\frac{2GM}{c^2 r} < 1$ can be written as $\frac{2GM}{c^2} < r$ where $\frac{2GM}{c^2}$ is the Schwarzschild radius of the gravitating object and therefore this condition means that r is greater than the Schwarzschild radius. However, we should note that since r is supposed to be coordinate variable (rather than physical variable) then the term "Schwarzschild radius" in such context is rather misleading and may not be accurate because "radius" suggests the normal meaning of radius rather than "coordinate radius". Accordingly, "Schwarzschild radius" is more appropriate as a classical term (although the "Schwarzschild" label may not be classical), while "Schwarzschild radial coordinate" should be more appropriate as a general relativistic term.
2. What you note about the formula $d\tau = \left(1 - \frac{2GM}{c^2 r}\right)^{1/2} dt$?
   Answer: For example, we note the following:
   • We input coordinate variables into the right hand side of this metrical relation and output a physical variable on the left hand side.
   • When we apply this relation in real physical situations we might be obliged to use physical variables instead of coordinate variables because in real life situations we have access only to physical variables,

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[137] This may have an analogy in the linearized general relativity where the gravitational effects can be treated as a physical field in a flat spacetime rather than a distortion of spacetime. In fact, this approach (i.e. treating any potential curvature of spacetime as a physical effect in a flat-spacetime background) should be facilitated by the fact that any such curvature should be very tiny.

e.g. what we know directly about a real physical event is its physical radial variable (which corresponds to r) and not its coordinate variable r. In fact, this note is not restricted to this equation but it should extend to any similar equation that involves the factor $\left(1 - \frac{2GM}{c^2 r}\right)$ or indeed any factor that contains coordinate variables. Hence, this should put a question mark on the non-metrical significance of coordinates in general relativity and if they are usable in practice or not (at least in some situations where obtaining coordinate quantities from physical quantities may not be possible).

6.3.2 Interpretation of Coordinate Time

1. Is it useful to consider events at infinity which cannot be reached?
Answer: Infinity in practical contexts means sufficiently far from the source of gravity such that the effects of gravity are negligible within the required approximations and allowed errors and uncertainties. Hence, infinity in this sense could be at a rather short distance away.

6.3.3 Gravitational Time Dilation

1. The statement: "the proper time interval of a frame that is deeper in a gravitational well is shorter than the corresponding proper time interval of a frame that is higher in the gravitational well" may be criticized by saying: you are comparing the time in two different local frames which have no common standard of calibration and hence the comparison is baseless. Discuss this issue.
Answer: It is obvious that any comparison (e.g. being shorter or longer) requires a common standard or reference of calibration (e.g. a common stick of specific length). Such a reference is easy to find when we have a global frame in which a single standard of calibration does exist. However, in general relativity we do not have such a global frame and hence we need to find a common standard of calibration "somewhere" in the spacetime. This "somewhere" is infinity where all local frames and observers in the spacetime agree on the time at infinity and hence they can use it as a standard for calibration. In fact, this is no more than using the coordinate time t as the standard for calibration, and this should be justified by the relation $d\tau_\infty = dt$ (see § 6.3.2).
To elaborate on this important point we can say: although the spacetime of general relativity is covered by local frames it still has global parameters[138] that can be used for calibration. These global parameters are represented by the coordinates (like the Schwarzschild coordinate time t). However, since the coordinate time has no metrical (or physical) significance its validity as a standard for calibrating physical quantities (i.e. physical times in local frames) at different locations may not be obvious. To close this gap we make a connection between the global coordinate time t (which is not physical) and the local proper time at infinity τ_∞ (which is physical) through the relation $d\tau_\infty = dt$ (or $\tau_\infty = t$ assuming a common origin). So, in essence we go to infinity to link the global coordinate time t to the local physical time τ_∞ through the relation $\tau_\infty = t$ and by this we make the physical time τ_∞ a "global" standard for calibration (although actually it is still local in its physical significance). Now, since τ_∞ is a physical time then it should be a valid physical "global" standard for calibration.[139]
We should remark that the use of infinity (instead of some other specific location in the spacetime) for global calibration is not a necessity but it is simple, intuitive and straightforward.[140] Another remark is that although frames in general relativity are local, we may consider the set of coordinate variables to be a basis for a global frame in contrast to the physical (or metrical variables) which are the basis for the local frames. Accordingly, we may claim that in general relativity we have two types of frame: global coordinate frame (which is unique) and local physical frames (which are many).

6.3.4 Gravitational Frequency Shift

1. Justify the following statement: "A wave that is ascending in a gravitational well should be red shifted

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[138] "Global parameters" here should mean that their quantitative values are agreed-upon globally.

[139] This elaboration is made for pedagogical purposes and hence it is not essential.

[140] Accordingly, calibration at infinity (or any other location) is not a necessity and hence each of the two frames can be calibrated by the other frame (following the above theoretical justification).

while a wave that is descending in a gravitational well should be blue shifted".
Answer: As compared to the coordinate time interval dt (which is equal to the proper time at infinity), the proper time interval of a frame that is deeper in a gravitational well is shorter than the corresponding proper time interval of a frame that is higher in the gravitational well (see § 6.3.3). Therefore, the periodic time of a wave that is ascending in a gravitational well lengthens and hence its frequency decreases (i.e. it is red shifted) while the periodic time of a wave that is descending in a gravitational well shortens and hence its frequency increases (i.e. it is blue shifted).
For more clarity, the above justification may be phrased rather differently by saying: what is actually shortened and lengthened is the unit of time by which the periodic time is measured. For example, if we have 10 cycles of a wave that descended from r_2 to r_1 ($r_2 > r_1$) where the size of the unit of time (say relative to infinity) at r_1 is twice the size of the unit of time at r_2 (due to time dilation) then these 10 cycles will be seen at r_1 to occur in a time interval that is half their time interval at r_2 and hence their frequency at r_1 will be seen as twice their frequency at r_2.

6.3.5 Comparison with Classical Mechanics and Special Relativity

1. Compare the time of occurrence and the time of observation in special relativity and in general relativity.
   Answer: In both theories we should have time of occurrence and time of observation where these times differ in general when the locations of occurrence and observation differ. However, in special relativity the frame and time are global for each observer and hence the time recorded at any location in a given frame belongs to the whole frame (and thus to the whole spacetime). Therefore, a localized observer at any location in this global frame can simply use this global time in his comparison between occurrence and observation. The situation in general relativity is more complicated since there is no such global time to be used for comparison and hence each localized observer should use a localized reference time for comparison. As we saw in § 6.3.2 and § 6.3.3, the time at infinity (which is equal to the coordinate time of spacetime) may be used as localized reference time for comparison.
2. Compare between time dilation in special relativity and time dilation in general relativity.
   Answer: We note the following:
   • Time dilation in special relativity is a function of speed, while time dilation in general relativity is a function of spacetime coordinates in gravitational field (and hence it is a function of the local metric properties).
   • In special relativity time in time dilation essentially belongs to the occurrence of events in two global inertial frames, while in general relativity time in time dilation essentially belongs to the occurrence of events at one location and observation of events at another location. In fact, this difference is based on the difference in the nature of frames in the two theories, i.e. global frames in special relativity and local frames in general relativity.

6.4 Length in Schwarzschild Spacetime

6.4.1 Relation between Spatial Coordinates and Proper Length

1. Determine if the factor $\left(1 - \frac{2GM}{c^2 r}\right)^{-1}$ changes (i.e. increases or decreases) in the same sense as r or in the opposite sense (assuming $\frac{2GM}{c^2 r} < 1$). What about the factor $\left(1 - \frac{2GM}{c^2 r}\right)^{-1/2}$?
   Answer: If r increases/decreases then $\frac{2GM}{c^2 r}$ decreases/increases and $\left(1 - \frac{2GM}{c^2 r}\right)$ increases/decreases and hence $\left(1 - \frac{2GM}{c^2 r}\right)^{-1}$ decreases/increases. This means that the factor $\left(1 - \frac{2GM}{c^2 r}\right)^{-1}$ changes in the opposite sense to r, i.e. $\left(1 - \frac{2GM}{c^2 r}\right)^{-1}$ is a decreasing function of r. This should also apply to $\left(1 - \frac{2GM}{c^2 r}\right)^{-1/2}$ since taking the square root does not affect the sense of inequality.
2. Comment on our statement: the r coordinate surfaces in the Schwarzschild spacetime are similar to their counterparts in a spherical coordinate system.
   Answer: They are similar in shape but not in size. In fact, we are not considering in this similarity

the difference in the radial distance due to the involvement of r in the above quadratic form. In other words, we are only considering the shape of the r coordinate surfaces regardless of the difference in size between these individual surfaces in the Schwarzschild spacetime and the corresponding surfaces in a Euclidean spherical coordinate system. In fact, gravity has an effect (through the involvement of r) on these surfaces (and hence gravity has an indirect effect on the embedded circumferential components) since the r coordinate surfaces (which are spheres) shrink or expand in one space relative to the other. So, the circumferential adjustment is dependent on the radial adjustment and hence if there is no radial adjustment there will be no circumferential adjustment. The reader is also advised to refer to problem 2 of § 8.3.

6.4.2 Interpretation of Spatial Coordinates

1. Use the spatial quadratic form of the Schwarzschild metric to interpret the spatial Schwarzschild coordinates.
   Answer: The spatial quadratic form of the Schwarzschild metric is given by:
   $$(ds)^2 = \left(1 - \frac{2GM}{c^2 r}\right)^{-1} (dr)^2 + r^2 (d\theta)^2 + r^2 \sin^2\theta\, (d\phi)^2$$
   This shows that the θ and ϕ in the Schwarzschild coordinates are the same as the θ and ϕ coordinates of spherical coordinate system while the radial variable in the quadratic form is modified by the factor $\left(1 - \frac{2GM}{c^2 r}\right)^{-1}$.
2. The Schwarzschild metric converges to the flat Lorentz metric at infinity. What is the significance of this on the meaning and interpretation of the Schwarzschild coordinates?
   Answer: The significance is that the Schwarzschild coordinates (specifically t and r) have their physical special relativistic meaning at infinity (i.e. they have the familiar Minkowskian metrical meaning there with t representing time and r representing length instead of being just labels for coordinates as it is the case in the other parts of the space).

6.4.3 Gravitational Length Contraction

1. Discuss how the issue of gravitational length contraction is dealt with in the literature of general relativity.
   Answer: The issue of gravitational length contraction is generally avoided in the literature of general relativity and hence it is not discussed (except by some and marginally). This rather bizarre situation motivated some to devise and elaborate theoretical and practical reasons for why gravitational length contraction should not be as natural and authentic as gravitational time dilation. However, these proposed reasons are generally excuses that do not stand serious scrutiny and analysis. We note that the real motive of this view (i.e. length contraction is not as natural and authentic as time dilation) and its justifications may be the lack of evidence in support of gravitational length contraction (see § 9.5) and the desire of the supporters of general relativity to protect the theory from potential challenge by this. In our view, the lack of evidence in support of gravitational length contraction may indicate that the alleged effect of gravitational time dilation may not be really a time dilation effect but it could be a demonstration of another effect that is based on a different physical principle (i.e. other than time dilation) such as the conservation of energy. This will be investigated and elucidated further in the future.

6.4.4 Comparison with Classical Mechanics and Special Relativity

1. Is there a similarity between length contraction in special relativity and in general relativity? If so, what?
   Answer: Yes, there is a similarity between the special relativistic length contraction which occurs only in the direction of motion (but not perpendicular to the direction of motion) and the general

relativistic length contraction (assuming a Schwarzschild metric) which occurs only in the direction of gravitational field[141] (but not perpendicular to the direction of gravitational field) as seen earlier where gravitational length contraction in the Schwarzschild spacetime occurs only in the radial component but not in the circumferential component. In fact, this should be consistent with the spirit of the equivalence principle where motion (in the form of acceleration) causes effects that are similar (or equivalent) to the effects of gravity. So, the similarity between the kinematical length contraction of special relativity which is caused by the uniform motion and takes place in the direction of motion and the gravitational length contraction of general relativity which is caused by the equivalent to accelerated motion (i.e. gravity) and takes place in the equivalent direction to motion is logical and intuitive.

2. Comment on the fact that physical effects like time dilation and length contraction occur in special relativity only across global frames (each of which covers the entire spacetime) while in general relativity they occur across local frames (each of which covers only part of the spacetime).
   Answer: This should make general relativity the truly "relativity" theory because in special relativity physical quantities are still absolute within each global frame although they are relative across these frames while in general relativity they are relative even across local frames in the spacetime (see § 10.1.8).

6.5 General Relativity and Absolute Frame

1. Discuss the proposition that general relativity should require (at least in some of its details) an absolute frame of reference because acceleration (which is supposed to be equivalent to gravity according to the equivalence principle) does not have sensible interpretation without such a frame of reference.
   Answer: There are many examples within the framework of general relativity about the necessity of the existence of an absolute frame if we have to make sense of this theory. Many of these examples are related to the weak form of the equivalence principle. For instance, the bending of light in accelerating frames and the "quasi-gravitational" fields generated by accelerated motion do not make sense unless we assume the existence of an external frame that should be absolute in a sense to logicalize the distinctive behavior (i.e. the bending of light or the generation of "quasi-gravitational" field) that is observed in the accelerating frame specifically. We should also mention in this context the absolute significance of rotation in spacetime as demonstrated by the Newton's bucket experiment (see § 6.5.1) because there is no reference in this experiment to any particular aggregate of matter and hence absolute frame should either be an intrinsic property of spacetime (and therefore it is defined with no reference to any matter) or be defined with reference to the overall distribution of matter in the Universe. As we discussed in B4, a reference to the overall distribution of matter does not make any real difference with regard to the necessity of absolute frame because what we are interested in is the physical spacetime (rather than a purely theoretical spacetime) and this spacetime can be legitimately determined in an absolute sense with reference to an absolute physical frame defined by the large scale distribution of matter in the Universe.
   We should remark that the presumption of the existence of matter and energy in the Universe does not invalidate the application of Lorentz mechanics (or special relativity) because the validity of Lorentz mechanics is not dependent on the validity or invalidity of general relativity (i.e. we may accept Lorentz mechanics and reject general relativity). Moreover, the applicability of Lorentz mechanics as a good approximation in local frames (which could have very large extension on astronomical and cosmological scales) does not depend on the absence of matter and energy in the Universe even if we accept general relativity as a valid gravitational theory.[142] In brief, Lorentz mechanics is not an idealization that does not exist in real physical world (as claimed by some) and this is obviously justified by the wide

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[141] For clarity, we use "gravitational field" in spite of being primarily a classical paradigm.

[142] In fact, if the large scale mass distribution is homogeneous and isotropic (as required by the cosmological principle) then there should be no location-dependent or direction-dependent gravitational effect due to this large scale distribution, and hence the situation will be a background gravitational field that is not felt as a gravity but as something else like inertiality (in accord with the Mach principle).

applicability of Lorentz mechanics in many scientific fields regardless of gravity and regardless of the validity or invalidity of general relativity.

2. Discuss the claim that absolute frame is irrelevant to general relativity because the motion in spacetime is explained by the concept of geodesics in spacetime.
   Answer: This claim (which can be found in the literature) is based on confusing the issue of cause with the issue of phenomenon. The concept of geodesic may explain the issue of phenomenon (i.e. we can describe the motion in spacetime by geodesics) but it does not explain the fundamental principles (which general relativity rests upon) or the causes of this particular behavior. As we discussed earlier, the equivalence of acceleration to gravity and the absolute significance of acceleration and rotation (as well as other examples) require in their theoretical foundations the concept of absolute frame and this cannot be provided or replaced by the concept of geodesic.
3. What is the significance of bending of light signals in accelerating frames with regard to the invariance of the speed of light and the necessity of absolute frame?
   Answer: If we accept the general relativistic claim about the invariance of the speed of light (locally at least), then we should repeat what we said in B4 that is the speed of light is frame independent but its velocity is frame dependent because the straight trajectory of light in one frame becomes a curved trajectory in another frame (i.e. the accelerating frame). As we discussed earlier, this should imply the existence of an absolute frame relative to which the light follows a certain trajectory and accordingly the trajectory in any other frame should be determined by this reference trajectory. Otherwise, why the trajectory of light should not be straight or curved in all frames (which is inline with its invariance properties) or follow patterns different to the above.
4. Discuss some of the implications of the existence of an absolute frame in the spacetime of general relativity.
   Answer: For example, this existence should lead to the conclusion that even a frame in a state of gravitational free fall can be an accelerating frame relative to an absolute frame and hence it could have gravitational effects (according to the equivalence principle). In other words, such a frame is not effectively-inertial despite its free fall because the free fall neutralizes the gravitational effect caused by the gravitating object but it does not affect the accelerating effects (which are equivalent to gravitational effects according to the equivalence principle) that are caused by the motion relative to the absolute frame.
   Similarly, a frame at rest relative to a source of gravity can be effectively inertial if its acceleration relative to absolute frame cancels the effect of gravity (which is supposed to be equivalent to acceleration).
5. Discuss the issue of absolute frame as an epistemological or philosophical issue and its relation to science from these perspectives.
   Answer: There are two aspects to the issue of the existence of absolute frame: philosophical and epistemological. The philosophical aspect is about the existence (or non-existence) of absolute frame "in reality" while the epistemological aspect is about the need (or no need) of the current formalism of physics (especially the relativity theories) to such a frame to have a sensible interpretation for this formalism. Our interest (which is the interest of science and scientists) is in the epistemological aspect. In other words, regardless of the existence (or non-existence) of absolute frame in reality we may need to propose and hypothesize the existence (or non-existence) to make sense of the current formalism of physics. Yes, if someone came with a different formalism or modified the current formalism then our position about this issue may require change to adapt with the new formalism. So in brief, the focus of interest in science should be on the epistemological rather than the philosophical issue and hence from this perspective there could be one epistemological position toward one formalism or theory and a different position toward another although the experimental facts should have the final say about the legitimacy and validity of the formalism and hence the epistemology (and this should lead to a unique epistemological position).
6. What is the significance of the conservation of momentum and angular momentum on the existence of absolute frame?
   Answer: In our view, these conservation principles cannot be rationally explained without the exis-

tence of an absolute spacetime (and hence an absolute frame of reference) because the conservation requires definite and absolute meaning of direction and orientation. This may also be extended to the conservation of energy in its relation to time. However, we should take into account the lack of global significance of these conservation principles according to general relativity (see § 7.9).

7. Is there physical evidence for the existence of absolute frame?
   Answer: We can claim that there are two types of evidence about the necessity of the existence of absolute frame:
   • Theoretically-based evidence represented by the existence of a class of privileged inertial frames (in which no effect of acceleration or rotation does exist and hence certain physical laws, such as Newton's laws, do apply). We may also include in this category other theoretical indicators such as those related to the conservation principles (as discussed in the previous exercise).
   • Indirect observational evidence such as the existence of Cosmic Microwave Background Radiation (CMBR) whose isotropy in a certain class of privileged frames indicates the existence of an underlying absolute frame. In fact, the existence of things like cosmic time, fundamental observers and co-moving frames and coordinates in modern cosmology is an implicit confession of the necessity of the existence of absolute frame (although it is not necessarily in the classical sense). The cosmological expansion of the Universe (according to modern cosmology) may also be seen as an indirect admission of the reality of space (and hence the logical sensibility of the existence of absolute frame). The Sagnac effect may also be seen as an indicator to the absoluteness of rotation and hence the existence of absolute frame.
   Note 1: we note that at the large scale of the entire Universe "cosmic time" can be regarded as a "proper time" rather than just a "coordinate time" because all locations in the Universe are equivalent (as demanded by the cosmological principle). In other words, we have no "locations at infinity" to distinguish any location from other locations. Accordingly, we should have a universal (or global or absolute) proper time. In fact, the rationale and logic of many cosmological arguments and formulations depend on accepting such universal coordinates (or absolute frame). So, the need for an absolute frame in general relativity (or at least in some of its applications like relativistic cosmology) is as urgent as the need in special relativity (or Lorentz mechanics). In fact, we can even claim that the cosmological principle itself implicitly requires an absolute frame in the background. In other words, the spatial homogeneity and isotropy of the Universe lacks any scientific meaning or significance unless we have an absolute frame in the background by which we can make sensible comparisons between locations and directions on such a huge and universal scale. So, the cosmological principle is scientifically meaningless without the presumption of the existence of a global absolute frame.
   Note 2: we may add to the above evidence for the existence of absolute frame the physical indications of the reality of space (or spacetime or space-time) as represented by the existence of "free space" quantities[143] like the permittivity and permeability of free space, i.e. ε_0 and μ_0. This may also be linked to the constancy (and even invariance) of the characteristic speed of light (regardless of the interpretation of the constancy and invariance) due to its link to these free space quantities through the famous relation $\varepsilon_0\mu_0c^2 = 1$ which can be obtained from Maxwell's equations. In fact, the constancy and invariance of c should depend (at least partly) on the constancy and invariance of ε_0 and μ_0 (or at least their combination) which may not be the case in general and hence there is still a space for theories that may question the constancy and invariance of c. However, this should not affect the significance of this evidence since the implication is mainly based on ε_0 and μ_0 being "free space" quantities and not on being invariants or constants. This should similarly apply to c since it will be a free space quantity (thanks to the relation $\varepsilon_0\mu_0c^2 = 1$) regardless of being constant/invariant or not.
   Note 3: the principle of metric gravity (see § 1.8.4) may also be seen to necessitate the existence of absolute frame because it requires the reality of spacetime due to its realistic properties such as being curved by the presence of matter and energy and embedding real physical content since gravity (which essentially is a physical agent) is part of the structure of the spacetime (see § 10.1.21).
8. Challenge general relativity on the basis of the existence and non-existence of absolute frame.
   Answer: We can challenge general relativity whether the theory does accept the existence of absolute

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[143] "Free space" quantities mean quantities that belong to free space itself (as if it is a real entity and a medium for interaction and transmission) and not to a material medium like water and air.

frame or not because:

- If it does accept absolute frame then it can be challenged by certain theoretical defects like inconsistency in the equivalence principle (since this principle ignores the effect of acceleration relative to the absolute frame in free fall).
- If it does not accept absolute frame then it can be challenged by other theoretical defects like inconsistency in special relativity (which has impact on general relativity). In fact, this will lead even to defect and inconsistency in the equivalence principle itself since acceleration (which is embedded in the essence of this principle) cannot then be defined realistically and sensibly in certain physical situations (e.g. in free space) to be equivalent to gravity.

9. Follow the logic of the equivalence principle to analyze a gravitational system in which the gravitating object is at rest or moving uniformly relative to absolute frame (i.e. the frame of the gravitating object is inertial) while the gravitated object falls freely.
   Answer: In fact, the answer should depend on what we mean by "the gravitated object falls freely" and if it is relative to the gravitating object or unconditionally. So:
   - If the gravitated object falls freely relative to the gravitating object (i.e. it falls in a fashion as if there is no absolute frame), then the fall is not really a free fall. The reason is that for such a fall the motion will annul the effect of gravity (caused by the presence of the gravitating object) but it will be accelerated relative to the absolute frame and this acceleration requires a force. Accordingly, the gravitated object is not moving freely.
   - If the gravitated object falls freely unconditionally (i.e. it moves under the effect of no force at all) then it is freely moving although it is not really falling freely with respect to the gravitating object. In this case, we should assume that the gravitational force (due to the presence of the gravitating object) is balanced by the inertial force (due to the presence of absolute frame).

   Note: we note that a propelling mechanism is required to realize the inertiality of the gravitating object. We should also note that the answer follows the logic of the equivalence principle (as requested by the question) and hence it may not be entirely logical. In fact, the purpose of this question is to probe certain theoretical possibilities (about the equivalence principle and absolute frame) which require attention and investigation. So, the question and its answer are mainly argumentative.
10. What is the significance of the existence of orbiting systems (e.g. solar system) on the absoluteness of rotation and the existence of absolute frame?
    Answer: We note that some gravitating systems collapse (i.e. the gravitating and gravitated objects move directly toward each other) while other gravitating systems are orbiting (i.e. the gravitating and gravitated objects move around each other). Now, since "collapsing" and "orbiting" are distinct phenomena that are independent of the observer, this distinction should indicate the absoluteness of rotation (or orbiting) and hence the existence of absolute frame.

6.5.1 Absolute Frame and Mach Principle

1. Describe the essence of Mach principle in a few words.
   Answer: Inertia is a manifestation of the large scale cosmic distribution of matter (where "inertia" here includes all direct and indirect inertial effects).
2. Consider two spheres S_1 and S_2 rotating relative to one another and assume that S_2 bulges at its equator, how do we explain this difference? According to Einstein "No answer can be admitted as epistemologically satisfactory, unless the reason given is an observable fact of experience ... Newtonian mechanics does not give a satisfactory answer to this question. It pronounces as follows: The laws of mechanics apply to the space R_1 in respect to which the body S_1 is at rest, but not to the space R_2 in respect to which the body S_2 is at rest. But the privileged space R_1 ... is a merely factitious cause, and not a thing that can be observed". Discuss this issue.
   Answer: We should note first that this thought experiment is essentially the same as the Newton's bucket experiment. Regardless of the philosophical issues that are attached to the Newtonian view, the existence of a unique absolute frame should provide a convincing scientific and epistemological answer, i.e. the bulging sphere is rotating relative to this unique frame (which is the frame of space R_1) while

the non-bulging sphere is at rest with respect to this unique frame. As an abstract philosophical or mathematical space the privileged space R_1 may be factitious but as a physical space (that is potentially defined by the large scale distribution of matter) it is real and observable.

3. Assess (in general terms) the claimed evidence in support and against Mach principle and its link to general relativity.
   Answer: In the literature of general relativity we can find many claims of theoretical and observational evidence in support and against Mach principle. Also, there are many theoretical analyses related to the link of this principle to the theory of general relativity and its consequences and predictions (e.g. frame dragging in certain physical situations). However, in our view most (if not all) these claims and analyses are arguable and can be challenged and reversed. The main reason for this is the lack of a universally accepted theoretical framework for such claims and analyses (due to differences in understanding, interpretation, etc.), and in the absence of such a universal theoretical reference any judgment or conclusion belongs to its owner and cannot compel any body else.

Chapter 7
Physics of General Relativity

7.1 Coordinates of Spacetime

7.2 Time Interval and Length

1. Why coordinate dependent quantities lose their global significance despite the ability to calibrate by local standards (e.g. at infinity)?
   Answer: In our view, this is because the standards of calibration themselves are local and hence even these standards have no global significance. In other words, to have a global significance we should have a global standards of calibration. Of course, this should not affect the sensibility of making comparisons using local standards of calibration (where these local standards may be "globalized" in some sense, e.g. by linking them to the coordinate variables as explained earlier).
2. Discuss, briefly, the physical effects that influence time interval and length in general relativity and compare them to those in special relativity.
   Answer: The main physical effects (assuming Schwarzschild metric) are gravitational time dilation and gravitational length contraction. These should correspond to the kinematical time dilation and kinematical length contraction of special relativity. We should remark that in this correspondence we view general relativity from the perspective of being a gravity theory (whose main objective is to determine the effect of the distribution of matter on the geometry of the spacetime and its metrical properties) rather than from the perspective of being a "General Theory" that incorporates even the formalism of special relativity. Otherwise, we should also have kinematical (as well as gravitational) time dilation and length contraction within this extended formalism of general relativity.

7.3 Frequency

1. Compare frequency in general relativity with frequency in special relativity and in classical physics.
   Answer: The basic concept of frequency (i.e. temporal repetition of a physical process) is the same in both relativity theories as well as in classical physics. However, the effects that affect frequency in these theories are different, i.e. Doppler frequency shift in special relativity (as well as in classical physics but with different formulation) and gravitational frequency shift in general relativity although general relativity (as a "General Theory") should also include Doppler frequency shift in its extended formalism.

7.4 Mass

1. Why mass in general relativity should be global while quantities like length and time interval should be local?
   Answer: The local nature of any physical quantity originates from its direct dependency on coordinates and local metric properties due to the curved nature of spacetime. Hence, quantities like length and time interval should be local because they are based directly on the spacetime (where length is related to space and time interval is related to time) and its metric properties. On the other hand, mass is not dependent on the spacetime and its metric properties (in fact mass according to general relativity contributes to the creation of these properties) and hence it should be global.
2. Compare mass in general relativity with mass in special relativity.
   Answer: Mass in general relativity is a common and unique concept, unlike mass in special relativity where (according to the old convention) we have proper (or rest) mass and improper (or non-rest) mass. In fact, this is from the perspective of general relativity as a gravity theory (rather than a "General Theory" that incorporates even the formalism of special relativity); otherwise we still have

proper mass and improper mass (according to that convention) even in general relativity.
Note: the distinction between proper and improper mass (according to the old convention) is based on kinematical (i.e. special relativistic) considerations. So, should we have a corresponding distinction based on gravitational (i.e. general relativistic) considerations (similar for example to time dilation which has gravitational as well as kinematical origin)?

3. Discuss the issue of the nature of mass in general relativity (as being local or global) from the perspective of proper and improper mass (assuming the validity of special relativity in the spacetime of general relativity).
Answer: Our view is that proper mass should be global according to the framework of general relativity, while improper mass should be local since it includes kinetic energy which is local.

7.5 Velocity, Speed and Acceleration

1. It may be claimed that mathematical constructions and techniques like parallel transport can give global significance to physical quantities that depend on coordinates and orientations. Discuss this claim briefly.
Answer: These mathematical constructions and techniques do not give global significance to the concerned quantities in a real physical global sense but they only add (at the best) mathematical (or formal or conventional) global significance to these quantities.[144] Moreover, it is not obvious that all locally defined quantities can be globalized by these mathematical constructions and techniques.
Note: the mathematical constructions and techniques (e.g. parallel transport) that allow comparison of physical quantities at different locations in curved spacetime are similar in a sense to the calibration at infinity as both allow a sort of comparison but they do not give a global physical significance to these quantities in a real and natural way since they are essentially of a mathematical (or formal or conventional) nature rather than a real physical nature. For example, energy as a real physical measure for the ability to do work in the real world is not just a number that can be defined and calibrated in a certain way to make it comparable in a certain sense.

7.6 Force

7.7 Momentum

7.8 Energy and Work

1. Give an example of the relativity of work in the spacetime of general relativity (e.g. Schwarzschild spacetime).
Answer: We may imagine a stationary person hanging a massive particle radially by a long massless inelastic string in the vicinity of a Schwarzschild object. Now, if the person pulls the string then (due to gravitational length contraction which affects the displacement at the location of the person and at the location of the particle as well as possible difference in the force of pulling at the two locations) the amount of work needed to displace the particle (at its location) is different from the amount of work spent by the person (at his location). However, the validity of this depends on several (rather ambiguous) aspects of the theory of general relativity (e.g. the interpretation of gravitational length contraction) and hence any analysis will not be conclusive (although it should be illuminating and beneficial from a pedagogical perspective).
Note: in fact, when we say above "the amount of work needed to displace the particle ... etc." we should have a single standard of calibration to compare the amount of work at the two locations. As long as the standards of calibration in general relativity are local, then the issue of the relativity of work (and the lack of global significance) in the spacetime of general relativity is still there even if we

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[144] Regarding parallel transport, the path dependency of the transported vector should be sufficient to demonstrate the lack of global and absolute significance. For example, by parallel-transporting a momentum vector over two different paths we get different vectors at the end point and hence there is no absolute and global meaning of the conservation of momentum (and even the momentum itself).

assume that the amount of work at the two locations is quantitatively equal (according to a certain local standard of calibration).

7.9 Conservation Principles

1. Discuss the principles of conservation of linear and angular momentum within the framework of general relativity.
   Answer: In our view the conservation of linear and angular momentum should lose their significance at global level in general relativity. One reason (among other reasons such as the relativity of time and length in the spacetime of general relativity) is that linear and angular momentum depend in their definition on the concepts of direction and orientation and these concepts lose their significance at global level because in curved space direction and orientation have only local significance. In fact, this should make some of the claimed evidence in support of general relativity (as well as the formulations which these evidence are based on) that depend on the conservation of momentum such as geodetic precession (see § 8.8 and § 9.8) questionable in their significance as endorsing tests to general relativity and its theoretical framework and rationale. Our belief is that linear and angular momentum, as well as their conservation principles, have global significance only within a uniform spacetime whose structural properties are independent of any local physical phenomenon like gravity (whether such a spacetime should be flat or not could be a matter of debate).
2. According to the relativity theories the 4-velocity (and hence the 4-momentum which is the 4-velocity times the supposedly-constant mass) of a free massive test object is a constant tangent vector to its world line which is a geodesic. Discuss this issue and speculate about its significance.
   Answer: We note the following:
   • The 4-momentum $\mathbf{P}$ of the object is parallel-transported along its world line (because the 4-momentum is tangent to a geodesic). This can be expressed mathematically as:

$$\frac{\delta \mathbf{P}}{\delta \lambda} = \mathbf{0}$$

where the affine parameter λ in this context represents proper time. This can be seen as a generalization of the Newton's first law that includes gravity (since gravity is incorporated in the geometry of spacetime and it is not a force). With this insight the relativity theories provide a bridge between Newton's laws of motion and Newton's law of gravity (although we should always be aware of the classical limits of these laws and the limit on any relativistic extension and connection). This issue must also be linked to the issue of geodesic equation where the geodesic equation with a vanishing force term (i.e. zero on the right hand side) imitates Newton's first law (including gravity), while the geodesic equation with a non-vanishing force term imitates Newton's second law (see exercise 8 of § 2.9.5). In fact, the above equation should be regarded as another form of the geodesic equation considering the relation between $\mathbf{P}$ and $\mathbf{U}$, i.e. $\mathbf{P} = m\mathbf{U}$ (see Problems of § 2.9.5). It should also be regarded as a link between Newton's laws and the conservation principles (noting that the temporal component of the 4-momentum represents energy divided by c while its spatial components represent 3-momentum).

• From the previous point it may be concluded that although the conservation principles have no global significance we may still be able to attach some global significance to them through parallel transport (for example) since the essence of parallel transport is based on comparing vectors "globally" at different locations. However, the rationality of this global significance should depend on certain assumptions about the meaning and interpretation of parallel transport of the 4-momentum in this context. For example, what is the meaning and significance of the constancy of mass in the absence of a global standard for energy (noting the mass-energy equivalence) where mass could be standardized and measured differently in different locations of spacetime. Also, what is the meaning and significance of the constancy (in magnitude and direction) of the parallel-transported 4-momentum in the context of this alleged global conservation (noting that parallel transport should have certain global significance since the essence of parallel transport is based on comparing vectors at different locations) where neither magnitude nor direction have global significance. In fact, this should be addressed within the formulation and interpretation of parallel transport in curved spaces in general and it is not specific to the present issue.

• If we extend the above ideas and arguments to massless objects (noting that massless objects have 4-momentum although they do not have rest mass) then we can extend the above "relativistic formulation of Newton's first law" to include massless objects although λ in this case should not represent proper time.

3. Give a technical meaning to the conservation of energy and momentum of free massive objects in the spacetime of general relativity.
   Answer: We can say that the 4-momentum of free objects (which follow geodesics) is parallel-transported along their geodesic trajectories. In other words, parallel transportation means conservation of the momentum 4-vector (since the momentum 4-vector is kept constant) and this conservation means conservation of energy and momentum (noting that the temporal component of the 4-momentum represents energy divided by c while its spatial components represent 3-momentum). In fact, this is the essence of the previous question.

7.10 Orbital Motion in Terms of Constants of Motion

1. Obtain Eq. 162 (in the text) from Eq. 266 (in the text).[145]
   Answer: We have:

$$\frac{dr}{d\tau} = \frac{dr}{d\phi}\frac{d\phi}{d\tau} = \frac{B}{r^2}\frac{dr}{d\phi}$$

where we used the chain rule in the first equality and Eq. 163 (in the text) in the second equality (with $u = 1/r$). On substituting from the last equation into Eq. 266 we get:

$$\begin{aligned}
\left(\frac{B}{r^2}\frac{dr}{d\phi}\right)^2 + \frac{B^2}{r^2}\left(1 - \frac{2GM}{c^2 r}\right) - \frac{2GM}{r} &= c^2\left(A^2 - 1\right) \\
\left(\frac{B}{r^2}\frac{dr}{d\phi}\right)^2 + \frac{B^2}{r^2} &= c^2\left(A^2 - 1\right) + \frac{2GMB^2}{c^2 r^3} + \frac{2GM}{r} \\
\left(\frac{1}{r^2}\frac{dr}{d\phi}\right)^2 + \frac{1}{r^2} &= \frac{c^2}{B^2}\left(A^2 - 1\right) + \frac{2GM}{c^2 r^3} + \frac{2GM}{rB^2} \\
\left(\frac{du}{d\phi}\right)^2 + u^2 &= \frac{c^2}{B^2}\left(A^2 - 1\right) + \frac{2GM}{c^2}u^3 + \frac{2GM}{B^2}u \qquad (18)
\end{aligned}$$

where in line 3 we divided by B^2 and in line 4 we used $u = 1/r$ noting that:

$$\left(\frac{du}{d\phi}\right)^2 = \left[\frac{d\left(1/r\right)}{d\phi}\right]^2 = \left(-\frac{1}{r^2}\frac{dr}{d\phi}\right)^2 = \left(\frac{1}{r^2}\frac{dr}{d\phi}\right)^2$$

On differentiating Eq. 18 with respect to ϕ we obtain:

$$\begin{aligned}
2\frac{du}{d\phi}\frac{d^2u}{d\phi^2} + 2u\frac{du}{d\phi} &= 0 + \frac{2GM}{c^2}3u^2\frac{du}{d\phi} + \frac{2GM}{B^2}\frac{du}{d\phi} \\
\frac{d^2u}{d\phi^2} + u &= \frac{3GMu^2}{c^2} + \frac{GM}{B^2}
\end{aligned}$$

where in the second line we divided by $2\frac{du}{d\phi} \neq 0$ (assuming non-circular trajectory). The last equation is the same as Eq. 162.

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[145] In fact, this should serve as verification for the derived energy equation.

2. Obtain Eq. 165 (in the text) from Eq. 272 (in the text).
   Answer: If we follow a similar method to the method of the previous question then we have:

$$\frac{dr}{d\lambda} = \frac{dr}{d\phi}\frac{d\phi}{d\lambda} = \frac{B}{r^2}\frac{dr}{d\phi}$$

   where we used Eq. 166 (in the text) in the second equality (with $u = 1/r$). On substituting from the last equation into Eq. 272 we get:

$$\begin{aligned}
\left(\frac{B}{r^2}\frac{dr}{d\phi}\right)^2 + \frac{B^2}{r^2}\left(1 - \frac{2GM}{c^2 r}\right) &= c^2A^2 \\
\left(\frac{B}{r^2}\frac{dr}{d\phi}\right)^2 + \frac{B^2}{r^2} &= c^2A^2 + \frac{2GMB^2}{c^2r^3} \\
\left(\frac{1}{r^2}\frac{dr}{d\phi}\right)^2 + \frac{1}{r^2} &= \frac{c^2A^2}{B^2} + \frac{2GM}{c^2r^3} \\
\left(\frac{du}{d\phi}\right)^2 + u^2 &= \frac{c^2A^2}{B^2} + \frac{2GM}{c^2}u^3 \qquad (19)
\end{aligned}$$

   where in the last equation we used $u = 1/r$ plus $\left(\frac{du}{d\phi}\right)^2 = \left(\frac{1}{r^2}\frac{dr}{d\phi}\right)^2$ which we obtained in the previous question. On differentiating Eq. 19 with respect to ϕ we obtain:

$$\begin{aligned}
2\frac{du}{d\phi}\frac{d^2u}{d\phi^2} + 2u\frac{du}{d\phi} &= 0 + \frac{2GM}{c^2}3u^2\frac{du}{d\phi} \\
\frac{d^2u}{d\phi^2} + u &= \frac{3GMu^2}{c^2}
\end{aligned}$$

   where in the second line we divided by $2\frac{du}{d\phi} \neq 0$ (assuming non-circular trajectory). The last equation is the same as Eq. 165.
3. Derive an equation for the radial motion (i.e. motion with constant θ and constant ϕ) of free massive object using Eq. 266 (in the text).
   Answer: Because the motion is radial the object has no angular momentum and hence $B \equiv J/m = 0$. Accordingly, Eq. 266 becomes:

$$\left(\frac{dr}{d\tau}\right)^2 = c^2\left(A^2 - 1\right) + \frac{2GM}{r} \qquad (20)$$

   On taking the τ-derivative of the last equation we obtain:

$$\begin{aligned}
2\left(\frac{dr}{d\tau}\right)\frac{d^2r}{d\tau^2} &= 0 - \frac{2GM}{r^2}\frac{dr}{d\tau} \\
\frac{d^2r}{d\tau^2} &= -\frac{GM}{r^2}
\end{aligned}$$

   where in the last line we divided by $2\left(\frac{dr}{d\tau}\right) \neq 0$. As we see, the last equation looks very similar to its Newtonian counterpart (which expresses the equality of the acceleration of the radial motion to the gravitational force field, i.e. Newton's second law combined with Newton's gravity law and the classical equivalence principle).[146] However, we should note that the derivative in the last equation is with respect to τ rather than t as in the Newtonian equation. Moreover, r in the last equation is the radial coordinate while r in the Newtonian equation is the radial distance. Nevertheless, in the classical limit we have $\tau \simeq t$ and $r \simeq R$ (where R is the radial distance) and hence the general relativistic formulation

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[146] In fact, the above equation is a scalar equation with the minus sign indicating the attractive nature of the gravitational force field.

converges to its classical counterpart.
Note: if we consider a particle of mass m dropped from rest from an initial radial coordinate r_i and it is freely falling radially toward $r < r_i$, then from Eq. 20 (noting that $\frac{dr}{d\tau} = 0$ initially) we get:

$$\begin{aligned} c^2\left(A^2 - 1\right) + \frac{2GM}{r_i} &= 0 \\ c^2\left(A^2 - 1\right) &= -\frac{2GM}{r_i} \end{aligned} \tag{21}$$

Hence, Eq. 20 can be rewritten as:

$$\begin{aligned} \left(\frac{dr}{d\tau}\right)^2 &= \frac{2GM}{r} - \frac{2GM}{r_i} \\ \frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 &= \frac{GmM}{r} - \frac{GmM}{r_i} \\ \frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 - \frac{GmM}{r} &= -\frac{GmM}{r_i} \end{aligned}$$

which is similar in form to the Newtonian formula that represents the relation between the kinetic and potential energy in a gravitational field, i.e. loss/gain in kinetic energy leads to gain/loss in potential energy and vice versa due to the conservation of total energy (which is negative because the state is bound).[147] However, we should note again the difference between the two formulations in the τ and t derivatives as well as in the meaning of physical and coordinate variables. We should also note that the conservation of energy has no global significance in general relativity. Moreover, things like “gravitational potential energy” are essentially classical (rather than relativistic) paradigms.

7.11 Effective Potential in Orbital Motion

1. Discuss briefly gravitational potential energy in general relativity.
   Answer: Strictly speaking, there is no meaning of gravitational potential energy in general relativity due to the paradigm of geodesic which replaces the classical paradigm of force. In fact, this may also be inline with the position of general relativity toward energy in general where it loses its significance at global level as discussed earlier.
2. Use the equivalence principle to argue that gravitational potential energy has no real meaning in general relativity.
   Answer: According to the equivalence principle the gravitational field can be annulled by free fall and hence any presumed “gravitational potential energy” will cease to exist (and thus it has no real meaning). However, this may be challenged because even if the gravitational field and potential energy vanish in the freely falling frame they can still exist in other frames.

7.12 Radial Trajectories in Spacetime

1. Show that Eq. 282 (in the text) is obtained by integrating Eq. 281 (in the text).
   Answer: Instead of integrating Eq. 281 to obtain Eq. 282, we differentiate Eq. 282 with respect to r to obtain Eq. 281, that is:

$$c\frac{dt}{dr} = \pm\left(1 + \frac{R_S}{r - R_S}\right) = \pm\left(\frac{r - R_S + R_S}{r - R_S}\right) = \pm\left(\frac{r}{r - R_S}\right) = \pm\left(1 - \frac{R_S}{r}\right)^{-1}$$

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[147] This should explain why $A^2 < 1$ in this case, as can be seen from Eq. 21. However, there seems to be an inconsistency about the meaning of the total energy E in the general relativistic and classical formulations. This should affect the sign of E and A and their size. In fact, there are many details here that we do not discuss (due to lack of priority and space). Anyway, the note in this question is mainly for pedagogical purposes and hence it can be ignored.

2. Find an analytical expression for the proper time (as a function of r) for a massive object dropped from rest at an initial radial coordinate r_i far away from the source of gravity (and hence it is freely falling radially toward the gravitating object).
Answer: Assuming Schwarzschild geometry (in Schwarzschild coordinates), we use Eq. 271 (in the text). Because the object is dropped from rest and it moves radially, we have $E^2/\left(m^2c^4\right) = 1$ and $J = 0$ and hence Eq. 271 becomes:

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{2GM}{r} \qquad \text{that is} \qquad \frac{dr}{d\tau} = -\sqrt{\frac{2GM}{r}}$$

where we take the negative root because the object is falling toward the gravitating object (i.e. r decreases as τ increases). Hence:

$$\begin{aligned} d\tau &= -\sqrt{\frac{r}{2GM}}dr \\ \tau &= -\frac{2}{3}\frac{r^{3/2}}{\sqrt{2GM}} + C \end{aligned}$$

Now, if we take $\tau = 0$ at $r = r_i$ then $C = \frac{2}{3}\frac{r_i^{3/2}}{\sqrt{2GM}}$ and hence the last equation becomes:

$$\tau = \frac{2}{3}\frac{r_i^{3/2}}{\sqrt{2GM}} - \frac{2}{3}\frac{r^{3/2}}{\sqrt{2GM}} = \frac{2}{3\sqrt{2GM}}\left(r_i^{3/2} - r^{3/2}\right) \tag{22}$$

Because $r_i \geq r$ we have $\tau \geq 0$ as it should be.

3. What is the 4-velocity of the object in the previous exercise?
Answer: The 4-velocity in the Schwarzschild coordinates (noting that the θ and ϕ components are zero because the object is moving radially) is given by:

$$\left(\frac{dx^0}{d\tau}, \frac{dx^1}{d\tau}, \frac{dx^2}{d\tau}, \frac{dx^3}{d\tau}\right) = \left(c\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, 0\right)$$

Now, from Eq. 161 in the text (noting that the radial trajectory is a geodesic, moreover $F = 1 - \frac{2GM}{c^2r}$ and in our case $A = \frac{E}{mc^2} = 1$) we obtain $\frac{dt}{d\tau} = \left[1 - \frac{2GM}{c^2r}\right]^{-1}$ and from the previous exercise we have $\frac{dr}{d\tau} = -\sqrt{\frac{2GM}{r}}$. Therefore, the 4-velocity of the object is:

$$\left(c\left[1 - \frac{2GM}{c^2r}\right]^{-1}, -\sqrt{\frac{2GM}{r}}, 0, 0\right) = \left(c\left[1 - \frac{R_S}{r}\right]^{-1}, -\sqrt{\frac{c^2R_S}{r}}, 0, 0\right)$$

4. Find an analytical expression for the coordinate time (as a function of r) for a massive object dropped from rest at an initial radial coordinate r_i far away from the source of gravity (and hence it is freely falling radially toward the gravitating object).
Answer: Assuming Schwarzschild geometry (in Schwarzschild coordinates), we have:

$$\frac{dt}{dr} = \frac{dt}{d\tau}\frac{d\tau}{dr} = -\left(1 - \frac{R_S}{r}\right)^{-1}\left(\frac{c^2R_S}{r}\right)^{-1/2} = -\left[\left(1 - \frac{R_S}{r}\right)\left(\frac{c^2R_S}{r}\right)^{1/2}\right]^{-1} \tag{23}$$

where in the first equality we use the chain rule and in the second equality we use the results of the previous exercise. On integrating the last expression we obtain:[148]

$$t = \int -\left[\left(1 - \frac{R_S}{r}\right)\left(\frac{c^2R_S}{r}\right)^{1/2}\right]^{-1} dr$$

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[148] The integration is performed using symbolic algebraic software (at www.integral-calculator.com) and the result is found (after simplification) to be identical to similar results found in the literature. The result was also checked by differentiating Eq. 24 to obtain Eq. 23.

$$= \quad -2\left[\frac{r^{3/2}+3R_S\sqrt{r}}{3c\sqrt{R_S}}+\frac{R_S}{2c}\ln\left(\frac{\sqrt{r}-\sqrt{R_S}}{\sqrt{r}+\sqrt{R_S}}\right)\right]+C \tag{24}$$

where C is the constant of integration. If $t=0$ at $r=r_i$ then:

$$C = 2\left[\frac{r_i^{3/2}+3R_S\sqrt{r_i}}{3c\sqrt{R_S}}+\frac{R_S}{2c}\ln\left(\frac{\sqrt{r_i}-\sqrt{R_S}}{\sqrt{r_i}+\sqrt{R_S}}\right)\right]$$

Note: on comparing Eq. 22 with Eq. 24 we can see that a massive object falling freely radially toward a black hole requires finite proper time but infinite coordinate time to reach the event horizon (noting that coordinate time represents the proper time of an observer at infinity).

7.13 Geodesic Deviation

1. Analyze the equation of geodesic deviation (i.e. Eq. 284 in the text) in flat spacetime.
   Answer: In flat spacetime $R^{\alpha}_{\beta\mu\gamma}=0$ and hence the equation of geodesic deviation reduces to $\frac{\delta^2 D^\mu}{\delta\lambda^2}\equiv\frac{d^2 D^\mu}{d\lambda^2}=0$ (assuming Cartesian coordinates) whose solution is $D^\mu=\alpha^\mu\lambda+\beta^\mu$ where α^μ and β^μ are constant 4-vectors. In fact, the equation $\frac{d^2D^\mu}{d\lambda^2}=0$ effectively means zero relative acceleration. Due to the tensorial nature of the solution (as well as the equation) this applies even to the general coordinates form, i.e. $\frac{\delta^2 D^\mu}{\delta\lambda^2}=0$. This can also be seen clearly from the alternative forms of the equation of geodesic deviation, e.g. $\frac{\delta^2 D^\alpha}{\delta\lambda^2}=R^{\alpha}_{\beta\gamma\mu}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda}D^\mu$ which clearly shows that the relative acceleration $\frac{\delta^2 D^\alpha}{\delta\lambda^2}$ vanishes when the spacetime is flat, i.e. when $R^{\alpha}_{\beta\gamma\mu}=0$.
2. Derive the equation of geodesic deviation (i.e. Eq. 284 in the text).
   Answer: We use in this derivation the assumptions, symbols and physical description (including Figure 3) that we used in the text. However, before we start derivation we need to clarify an important point that is: what sort of equation (or mathematical condition) we are looking for to describe and quantify geodesic deviation? As indicated in the text, geodesic deviation essentially describes and quantifies the relative acceleration between two freely falling particles along two adjacent geodesics. So, simplistically D^μ represents "displacement", its first (absolute) derivative represents "relative velocity" and its second (absolute) derivative represents "relative acceleration" (which is supposedly the essence of geodesic deviation). The requirement of *absolute* derivative is obvious because we are measuring the variation along curves (i.e. geodesics) in a generally curved spacetime using general coordinates. Accordingly, we should be looking for a formula (or mathematical expression) for the second order absolute derivative $\frac{\delta^2 D^\mu}{\delta\lambda^2}$ of the displacement 4-vector D^μ.
   Now, to simplify the derivation let use geodesic coordinates at point $\mathbf{x}_1$ and hence the Christoffel symbols at $\mathbf{x}_1$ vanish although their derivatives may not (see the upcoming note 1). Since $\mathbf{x}_1$ (which is the pole of geodesic coordinates) is a point on the geodesic C_1 then the geodesic equation of C_1 at $\mathbf{x}_1$ is given by:[149]

$$\frac{d^2x_1^\alpha}{d\lambda^2}=0 \tag{25}$$

where the subscript "1" is a reference to $\mathbf{x}_1$ and where the vanishing of the Christoffel symbols at the pole $\mathbf{x}_1$ is used. Similarly, $\mathbf{x}_2$ is a point on the geodesic C_2 and hence the geodesic equation of C_2 at $\mathbf{x}_2$ is given by:

$$\frac{d^2x_2^\alpha}{d\lambda^2}+{}_2\Gamma^{\alpha}_{\beta\gamma}\frac{dx_2^\beta}{d\lambda}\frac{dx_2^\gamma}{d\lambda}=0 \tag{26}$$

where the subscript "2" (including the one that precedes $\Gamma^{\alpha}_{\beta\gamma}$) is a reference to $\mathbf{x}_2$. As a first order approximation (using truncated Taylor series), the Christoffel symbols at $\mathbf{x}_2$ may be given by:

$${}_2\Gamma^{\alpha}_{\beta\gamma}={}_1\Gamma^{\alpha}_{\beta\gamma}+\left(\partial_\mu\Gamma^{\alpha}_{\beta\gamma}\right)_1\left(x_2^\mu-x_1^\mu\right)=\left(\partial_\mu\Gamma^{\alpha}_{\beta\gamma}\right)_1 D^\mu \tag{27}$$

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[149] We remind the reader that the geodesic equation is $\frac{d^2x^\alpha}{d\lambda^2}+\Gamma^{\alpha}_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda}=0$.

where $\left(\partial_\mu \Gamma^\alpha_{\beta\gamma}\right)_1$ represents the derivative of $\Gamma^\alpha_{\beta\gamma}$ at $\mathbf{x}_1$. We note that in the second equality the vanishing of the Christoffel symbols at $\mathbf{x}_1$ (i.e. ${}_1\Gamma^\alpha_{\beta\gamma} = 0$) and $x_2^\mu - x_1^\mu = D^\mu$ are used. On subtracting Eq. 25 from Eq. 26 we get:

$$\begin{aligned}
\left(\frac{d^2 x_2^\alpha}{d\lambda^2} - \frac{d^2 x_1^\alpha}{d\lambda^2}\right) + {}_2\Gamma^\alpha_{\beta\gamma}\frac{dx_2^\beta}{d\lambda}\frac{dx_2^\gamma}{d\lambda} &= 0 \\
\frac{d^2}{d\lambda^2}\left(x_2^\alpha - x_1^\alpha\right) + {}_2\Gamma^\alpha_{\beta\gamma}\frac{dx_2^\beta}{d\lambda}\frac{dx_2^\gamma}{d\lambda} &= 0 \\
\frac{d^2 D^\alpha}{d\lambda^2} + {}_2\Gamma^\alpha_{\beta\gamma}\frac{dx_2^\beta}{d\lambda}\frac{dx_2^\gamma}{d\lambda} &= 0 \\
\frac{d^2 D^\alpha}{d\lambda^2} + \left(\partial_\mu \Gamma^\alpha_{\beta\gamma}\right)_1 D^\mu \frac{dx_1^\beta}{d\lambda}\frac{dx_1^\gamma}{d\lambda} &= 0 \qquad (28)
\end{aligned}$$

where in line 3 we use $x_2^\alpha - x_1^\alpha = D^\alpha$ and in line 4 we substitute from Eq. 27 and replace the subscripts "2" (assuming $dx_2^\beta/d\lambda = dx_1^\beta/d\lambda$ and $dx_2^\gamma/d\lambda = dx_1^\gamma/d\lambda$ which should be inline with the use of a single natural or affine parameter λ on neighboring geodesics). Now, the second order absolute derivative of D^α at $\mathbf{x}_1$ is given by (where we drop the subscript "1" for simplicity and clarity):

$$\begin{aligned}
\frac{\delta^2 D^\alpha}{\delta\lambda^2} &\equiv \frac{\delta}{\delta\lambda}\left(\frac{\delta D^\alpha}{\delta\lambda}\right) \\
&= \frac{d}{d\lambda}\left(\frac{\delta D^\alpha}{\delta\lambda}\right) + \left(\frac{\delta D^\gamma}{\delta\lambda}\right)\Gamma^\alpha_{\gamma\beta}\frac{dx^\beta}{d\lambda} \\
&= \frac{d}{d\lambda}\left(\frac{\delta D^\alpha}{\delta\lambda}\right) \\
&= \frac{d}{d\lambda}\left(\frac{dD^\alpha}{d\lambda} + D^\gamma \Gamma^\alpha_{\gamma\beta}\frac{dx^\beta}{d\lambda}\right) \\
&= \frac{d^2 D^\alpha}{d\lambda^2} + \frac{d}{d\lambda}\left(D^\gamma \Gamma^\alpha_{\gamma\beta}\frac{dx^\beta}{d\lambda}\right) \\
&= \frac{d^2 D^\alpha}{d\lambda^2} + \frac{d}{d\lambda}\left(\Gamma^\alpha_{\gamma\beta}\left[D^\gamma \frac{dx^\beta}{d\lambda}\right]\right) \\
&= \frac{d^2 D^\alpha}{d\lambda^2} + \frac{d\Gamma^\alpha_{\gamma\beta}}{d\lambda}\left[D^\gamma \frac{dx^\beta}{d\lambda}\right] + \Gamma^\alpha_{\gamma\beta}\frac{d}{d\lambda}\left[D^\gamma \frac{dx^\beta}{d\lambda}\right] \\
&= \frac{d^2 D^\alpha}{d\lambda^2} + \frac{d\Gamma^\alpha_{\gamma\beta}}{d\lambda}\left[D^\gamma \frac{dx^\beta}{d\lambda}\right] \\
&= \frac{d^2 D^\alpha}{d\lambda^2} + \partial_\omega \Gamma^\alpha_{\gamma\beta} D^\gamma \frac{dx^\beta}{d\lambda}\frac{dx^\omega}{d\lambda}
\end{aligned}$$

where in line 2 we absolute-differentiate the vector $\frac{\delta D^\alpha}{\delta\lambda}$, in line 3 we use the fact that $\Gamma^\alpha_{\gamma\beta} = 0$ at $\mathbf{x}_1$, in line 4 we absolute-differentiate the vector D^α (keeping $\Gamma^\alpha_{\gamma\beta}$ since we are going to take its derivative which is generally not zero), in line 5 we use the sum rule of differentiation, in line 6 we group symbols, in line 7 we use the product rule of differentiation, in line 8 we eliminate the last term since $\Gamma^\alpha_{\gamma\beta} = 0$ at $\mathbf{x}_1$, and in line 9 we use the chain rule. On relabeling some dummy indices and reordering, the equation of line 9 can be written as:

$$\frac{\delta^2 D^\alpha}{\delta\lambda^2} - \frac{d^2 D^\alpha}{d\lambda^2} - \partial_\gamma \Gamma^\alpha_{\mu\beta} D^\mu \frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0 \qquad (29)$$

On adding the two sides of Eq. 28 (with dropping the subscripts "1") to the two sides of Eq. 29 we get:

$$\frac{\delta^2 D^\alpha}{\delta\lambda^2} - \partial_\gamma \Gamma^\alpha_{\mu\beta} D^\mu \frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} + \partial_\mu \Gamma^\alpha_{\beta\gamma} D^\mu \frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} \quad = \quad 0$$

$$\frac{\delta^2 D^\alpha}{\delta\lambda^2} + \left(\partial_\mu \Gamma^\alpha_{\beta\gamma} - \partial_\gamma \Gamma^\alpha_{\mu\beta}\right) D^\mu \frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} \quad = \quad 0 \tag{30}$$

Now, the Riemann-Christoffel curvature tensor is given by (see Eq. 77 in the text):

$$R^\alpha{}_{\beta\mu\gamma} = \partial_\mu \Gamma^\alpha_{\beta\gamma} - \partial_\gamma \Gamma^\alpha_{\beta\mu} + \Gamma^\rho_{\beta\gamma}\Gamma^\alpha_{\rho\mu} - \Gamma^\rho_{\beta\mu}\Gamma^\alpha_{\rho\gamma}$$

However, at $\mathbf{x}_1$ the Christoffel symbols vanish and hence we get:

$$R^\alpha{}_{\beta\mu\gamma} = \partial_\mu \Gamma^\alpha_{\beta\gamma} - \partial_\gamma \Gamma^\alpha_{\beta\mu}$$

Accordingly, Eq. 30 can be written as (noting the symmetry of $\Gamma^\alpha_{\beta\mu}$ in its paired indices):

$$\frac{\delta^2 D^\alpha}{\delta\lambda^2} + R^\alpha{}_{\beta\mu\gamma} D^\mu \frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0$$

which is the equation of geodesic deviation as given in the text. Finally, since $\mathbf{x}_1$ is an arbitrary point on an arbitrary geodesic curve the derived equation has general validity from these perspectives (i.e. it applies to any point on any geodesic).

Note 1: it may be thought that the vanishing of a function implies the vanishing of its derivative, e.g. $\frac{d0}{dt} = 0$.[150] However, this is not true in general because a function and its derivative are two different functions and the vanishing of one of these functions at a given point does not imply the vanishing of the other function at that point (e.g. the function $y = x - 1$ is zero at the point $x = 1$ but its derivative is not zero at that point since $\frac{dy}{dx} = 1 \neq 0$). In fact, we should distinguish between two cases: identical vanishing of a function over its domain (or over a continuous interval of its domain) and accidental vanishing of a function at a given point in its domain where in the first case the derivative should also vanish (i.e. zero function implies zero derivative) since the function is constant while in the second case the derivative should not vanish in general (i.e. zero function does not imply zero derivative) since the function is not constant and the function and its derivative are two different functions where each can accidentally vanish independent of the other.

Note 2: the equation of geodesic deviation facilitates the calculation of the rate of convergence (or divergence) of two adjacent geodesic curves as we move in one direction or the other along the curves.

Note 3: there are limitations in the above derivation (as well as other similar derivations) of the equation of geodesic deviation and this could lead to limitations and restrictions on its validity and applicability. For example, the derivation is based on assuming close adjacency between the two geodesics (and hence its validity could be restricted to the immediate neighborhood of a point on a geodesic) as well as using certain approximations. Moreover, parameterizing the two geodesics with the same natural (or affine) parameter and manipulating some symbols and twisting their significance could also impose restrictions on its applicability and affect its validity.

3. The general relativistic paradigm of geodesic deviation is commonly considered in the literature to be a substitute for the classical paradigms of gravitational gradient and tidal force. Comment on this.

   Answer: We believe that the paradigms of gravitational gradient and tidal force have dynamical contents that do not exist in the paradigm of geodesic deviation and this is reflected even in the application of the formalism of these paradigms. Accordingly, geodesic deviation does not comprehensively substitute these classical paradigms conceptually or practically. We should also note that the paradigms of gravitational gradient and tidal force have global significance while the paradigm of geodesic deviation does not due to the local nature of the affine parameter (which is used in the formalism of geodesic deviation) in the curved spacetime of general relativity. The similarity between geodesic deviation and acceleration (which is obviously local) should also support the locality of geodesic deviation.

   Note: from a more general perspective (i.e. more general than the perspective of comparison with tidal force and gravitational gradient), the formalism of geodesic deviation may be criticized for being restricted since it legitimately applies only to neighboring geodesics (which should justify their parameterization by a single affine parameter). This may also be expressed by saying: the displacement 4-vector $D^\mu(\lambda)$ must be local (or infinitesimal) because in curved spacetime displacement cannot extend legitimately over non-local expansion of spacetime (due to the curved nature of spacetime).

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[150] The example $\frac{d0}{dt} = 0$ reflects confusion between differentiating a function then evaluating it, and evaluating then differentiating. So, if we have a function $f(x)$ and it vanishes at $x = x_0$ [i.e. $f(x = x_0) = 0$] then what we should do to obtain its derivative at $x = x_0$ is to differentiate f first and then evaluate its derivative df/dx at $x = x_0$ rather than evaluating f at $x = x_0$ first then differentiating its value (which is meaningless).

Chapter 8
Consequences and Predictions

8.1 Perihelion Precession of Mercury

1. What "perihelion" and "aphelion" mean?
   Answer: Perihelion is the point in the planetary orbit at which the planet becomes closest to the Sun while aphelion is the point in the planetary orbit at which the planet becomes farthest from the Sun.
2. What "precession" means? Is it appropriate term for describing the slow shift in the position of perihelion?
   Answer: The term "precession" originally means the gradual reorientation of the rotational axis of a spinning object where this axis rotates around another axis due to the existence of a driving torque, as seen for example in spinning tops and gyroscopes. The similarity between this meaning and the gradual shift of the long axis of the orbit (which causes the shift in the perihelion position) should be obvious and this justifies the use of this term. The precession of perihelion is also commonly known as the advance of perihelion and this term may be more appropriate and straightforward in depicting this phenomenon. The precession may also be ascribed to the orbit (and hence we say for example "the precession of the orbit of Mercury") which may also be more appropriate in describing this phenomenon.
3. What we mean by "extra precession"?
   Answer: "Extra precession" is used in this book to refer to the precession that is not accounted for classically (at least within the mainstream classical physics, i.e. excluding classical theories like the retarded gravitational potential theory) and distinguish it from the precession caused for example by interaction with other planets. In fact, we could call this extra precession "relativistic precession" (or similar names that can be found in the literature) but we prefer to use "extra precession" because we do not want to restrict it to the relativity theory since it may be explained by a classical or non-classical theory other than the relativity theory.
4. What are the main causes of the precession of perihelion of Mercury that can be accounted for by classical physics?[151]
   Answer: The main causes that can be explained classically are:[152]
   • Precession of equinoxes in the geocentric coordinate system.[153] Errors in measurements (due to the fact that the Earth is not an inertial frame of reference although it is approximately inertial) may also be added.
   • Perturbations caused by gravitational interaction with other planets in the solar system.
5. What distinguishes Mercury from other planets in the solar system and hence makes it prominent with regard to its extra precession of perihelion?
   Answer: As indicated in the text, all planets should undergo extra precession of perihelion and hence Mercury is not unique in this regard. What distinguishes Mercury, however, is its large planetary factor (which makes its extra precession per revolution larger; see $\delta\phi$ column in Table 1 in the text) and its short orbital period (which makes the number of revolutions per century larger; see T and N_c columns in Table 1 in the text) and hence its extra precession per century becomes larger (see $\delta\phi_c$ column in Table 1 in the text).
6. What you note about $\delta\phi$, $\delta\phi_c$ and N_c in Table 1 (in the text)?
   Answer: They all decrease according to the order of the planets in their proximity to the Sun.

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[151] As indicated earlier, we are excluding here classical theories that claim to account for the extra precession.

[152] In fact, these causes can be classified into two main categories: those due to the choice of coordinate system and those due to gravitational interaction with other planets.

[153] Noting that the precession of equinoxes has a period of about 26000 years, a precession of about 50 arcseconds per year (i.e. about 5000 arcseconds per century) is attributed to this cause.

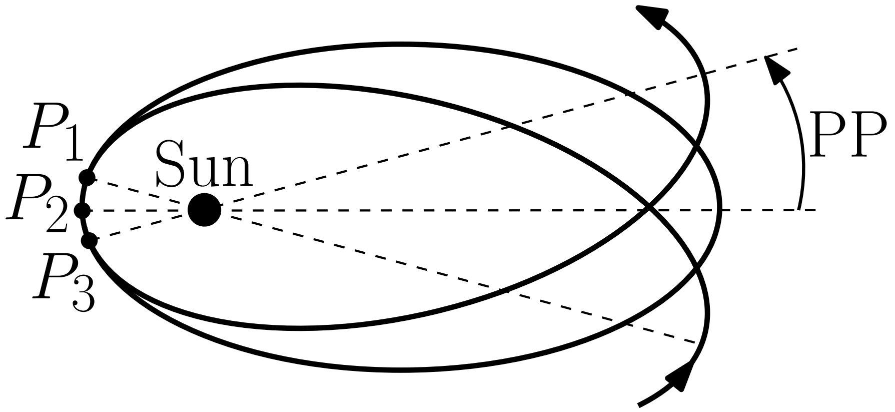

Figure 3: A simple graphical illustration of the precession of perihelion PP where P_1, P_2, P_3 represent successive perihelia. We note that the indicated precession can have various causes (and hence we use PP instead of $\delta\phi$).

7. Illustrate the precession of perihelion by a simple sketch.
   Answer: See Figure 3.
8. As we noted earlier (see § 1.1), the extra precession of perihelion of Mercury may be seen as an exception to the common rule that the characteristic general relativistic effects that distinguish general relativity from classical gravity are generally observable in strong gravitational fields. Discuss this issue.
   Answer: In our opinion, instead of viewing this as an exception to that rule it is more appropriate to see it as demonstration of the accumulation of a very tiny relativistic effect (and hence it may be seen as violation to the locality in time) where the effect becomes observable over a long period of time (i.e. century or even periodic time) and therefore the classical and general relativistic predictions are indistinguishable over a short period of time. In fact, the violation of the locality condition should extend even to space because of the spatial extension of the planetary orbit.

8.2 Light Bending by Gravity

1. Is light bending by gravity restricted to light (i.e. photons)?
   Answer: No, "light" here represents (according to general relativity and modern physics) any massless object. This should be clear from the derivation in the Problems where we used the geodesic equation of massless objects without any restriction to light. So, "light" here is more general than the optical part of the electromagnetic spectrum and indeed more general than the electromagnetic radiation.[154]
   In this context, it is noteworthy that there are a number of observed phenomena and experiments that are based on gravitational bending of electromagnetic radiation from other parts of the electromagnetic spectrum such as radio waves (see exercise 5 of § 9.2).
2. Discuss the following statement: "The bending of light in gravitational fields may also be justified qualitatively by the equivalence principle since light bends in accelerating frames and hence it should also bend in gravitational frames".
   Answer: We note the following:
   • Light bending in accelerating frames requires justification in the first place since there seems to be no direct experimental or observational verification of this (at least in the time of formulating general relativity). Yes, it may be deduced from the principles of Lorentz mechanics (or special relativity) since the velocity of light is not invariant across inertial frames although its speed is invariant (see B4). Accordingly, accelerating frames are considered as composition of series of inertial frames (i.e.

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[154] This is from a pure theoretical relativistic perspective. Whether this is consistent with observations (or other theories) or not is a different matter (as it might be the case with neutrino for example and if it follows the relativistic rules of gravity or not).

instantaneous rest frames) where the deflection of light increases gradually as the speed of the frame increases.[155]

• We should remind the reader of our view that acceleration requires absolute frame if it should have real physical consequences and significance (see § 6.5) and this should be a factor in determining in which frame the light should bend and how. Also, light bending in one frame or the other (i.e. two frames in relative acceleration) should depend on other factors such as to which frame (if any) the source of light belongs. In brief, there are several factors that should be considered for determining if and how light should bend in any particular frame.

• We may also consider the issue of decelerating frames (which should be included in accelerating frames in the general sense of acceleration) and hence we may ask: to which gravity these frames should correspond (refer to § 1.8.2)?

• We may also consider the issue of rotating frames (which are a special type of accelerating frames in the most general sense of acceleration) and to which gravity these frames should correspond?

3. What is the significance of light bending by gravity?

   Answer: It may be claimed that since light is massless, its bending by gravity should support the general relativistic view that gravity is an effect of the spacetime curvature (since light follows geodesic paths which are geometric attributes) and not a force since the concept of force is based on the existence of gravitational mass (i.e. the mass of the gravitated object). So, since we have no gravitated mass then we should also have no gravitating mass (as such) and therefore gravity is an effect of the spacetime geometry (which is the result of the mass-energy distribution) rather than a force in a technical sense.

   However, the rationale of the above argument may be challenged by the fact that light should have effective mass (according to the mass-energy equivalence relation $E = mc^2$) although it may not have rest mass.[156] Moreover, light could have rest mass in a classical sense according to the classical corpuscular theory of light. So, the validity of the above argument depends on the claim that gravitational force is restricted to objects with non-vanishing rest mass (which is not obvious) plus the claim that light has no rest mass (which is commonly accepted in modern physics although this may not be the case according to the classical corpuscular theory of light). As we will see, the bending of light by gravity can be derived in classical physics (although it is quantitatively different to that of general relativity) and hence it is consistent with the theoretical framework of Newtonian gravity and its paradigms (such as force and gravitational field) despite potential reservations on some derivation methods.

4. Show that $u = \frac{\cos\phi}{d}$ is a solution to the equation $\frac{d^2u}{d\phi^2} + u = 0$.

   Answer: This can be easily shown by inserting $u = \frac{\cos\phi}{d}$ into the equation $\frac{d^2u}{d\phi^2} + u = 0$, that is:

$$\frac{d^2u}{d\phi^2} + u = -\frac{\cos\phi}{d} + \frac{\cos\phi}{d} = 0$$

5. Verify that (**a**) the polar equation $\frac{1}{r} = \frac{\cos\phi}{d}$ (with $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$) is an equation of a straight line parallel to the y axis, and (**b**) the polar equation $\frac{1}{r} = \frac{\sin\phi}{d}$ (with $0 < \phi < \pi$) is an equation of a straight line parallel to the x axis.

   Answer: The transformation equations between polar coordinates (i.e. r and ϕ) and Cartesian coordinates (i.e. x and y) are:

$$r = \sqrt{x^2 + y^2} \qquad \cos\phi = \frac{x}{\sqrt{x^2 + y^2}} \qquad \sin\phi = \frac{y}{\sqrt{x^2 + y^2}}$$

   (**a**) Transforming from polar coordinates to Cartesian coordinates we get:

$$\frac{1}{r} = \frac{\cos\phi}{d}$$

---

[155] However, we should note that it is not true in general that the observations in an accelerating frame are the same as the observations in the instantaneous (inertial) rest frame (see for example § 10.1.1).

[156] For massless objects like light we use $E = mc^2$ (instead of $E_0 = mc^2$) because they do not have rest energy (noting that this relation is supposed to be for effective mass).

$$\begin{aligned}\frac{1}{\sqrt{x^2+y^2}} &= \frac{x}{d\sqrt{x^2+y^2}}\\ x &= d\end{aligned}$$

which is the Cartesian equation of a straight line parallel to the y axis and intercepting the x axis at d.

(**b**) Transforming again from polar to Cartesian we get:

$$\begin{aligned}\frac{1}{r} &= \frac{\sin\phi}{d}\\ \frac{1}{\sqrt{x^2+y^2}} &= \frac{y}{d\sqrt{x^2+y^2}}\\ y &= d\end{aligned}$$

which is the Cartesian equation of a straight line parallel to the x axis and intercepting the y axis at d.

6. Verify that:

$$u = \frac{\cos\phi}{d} + \frac{GM\left(2-\cos^2\phi\right)}{c^2d^2} \tag{31}$$

is a solution to the following equation:

$$\frac{d^2u}{d\phi^2} + u = \frac{3GM\cos^2\phi}{c^2d^2} \tag{32}$$

Answer: We have:

$$\begin{aligned}\frac{du}{d\phi} &= -\frac{\sin\phi}{d} + \frac{2GM\cos\phi\sin\phi}{c^2d^2}\\ \frac{d^2u}{d\phi^2} &= -\frac{\cos\phi}{d} - \frac{2GM\sin^2\phi}{c^2d^2} + \frac{2GM\cos^2\phi}{c^2d^2} \qquad (33)\end{aligned}$$

On substituting from Eqs. 33 and 31 into the left hand side of Eq. 32 we get:

$$\begin{aligned}\frac{d^2u}{d\phi^2} + u &= -\frac{\cos\phi}{d} - \frac{2GM\sin^2\phi}{c^2d^2} + \frac{2GM\cos^2\phi}{c^2d^2} + \frac{\cos\phi}{d} + \frac{GM\left(2-\cos^2\phi\right)}{c^2d^2}\\ &= -\frac{2GM\sin^2\phi}{c^2d^2} + \frac{2GM\cos^2\phi}{c^2d^2} + \frac{2GM}{c^2d^2} - \frac{GM\cos^2\phi}{c^2d^2}\\ &= -\frac{2GM\left(1-\cos^2\phi\right)}{c^2d^2} + \frac{2GM\cos^2\phi}{c^2d^2} + \frac{2GM}{c^2d^2} - \frac{GM\cos^2\phi}{c^2d^2}\\ &= -\frac{2GM}{c^2d^2} + \frac{2GM\cos^2\phi}{c^2d^2} + \frac{2GM\cos^2\phi}{c^2d^2} + \frac{2GM}{c^2d^2} - \frac{GM\cos^2\phi}{c^2d^2}\\ &= \frac{3GM\cos^2\phi}{c^2d^2}\end{aligned}$$

which is the right hand side of Eq. 32 and hence Eq. 31 is a solution to Eq. 32.

7. Show that for a light ray grazing a typical star, $\frac{GM}{c^2d}\cos^2\phi$ is very small (i.e. close to zero).

Answer: Let use the Sun as a typical star. So, for a light ray grazing the Sun we have:

$$\frac{GM}{c^2d} \simeq \frac{6.674\times10^{-11}\times1.989\times10^{30}}{\left(3\times10^8\right)^2\times6.9551\times10^8} \simeq 2.121\times10^{-6}$$

Moreover, according to our setting (i.e. the light ray follows a path oriented along the y-axis orientation with very small deflection) $\phi \simeq \pi/2$ and hence $\cos^2\phi \simeq 0$. Accordingly, $\frac{GM}{c^2d}\cos^2\phi$ is very small.

Note: the smallness of $\frac{GM}{c^2d}$ can be verified more generally as follows:

$$\frac{GM}{c^2d} = \frac{1}{2}\frac{2GM}{c^2d} = \frac{1}{2}\frac{R_S}{d}$$

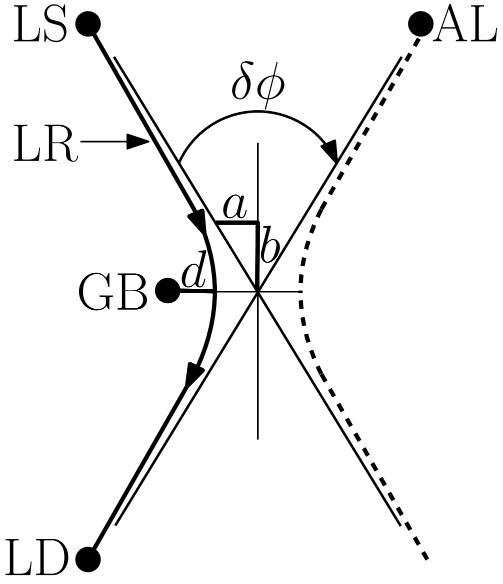

Figure 4: A simple sketch depicting the deflection of a light ray LR emitted from a light source LS as the ray passes by a gravitating body GB toward a light detector LD and hence the ray is deflected by an angle $\delta\phi$ causing the image of LS to be visually displaced and thus it is seen at an apparent location AL. The other symbols in the sketch are explained in the text.

where R_S is the Schwarzschild radius. Now, for ordinary stars $R_S \ll d$ and hence $\frac{GM}{c^2 d}$ is very small.

8. Derive a classical formula for light bending by stars (e.g. Sun) and hence show that the classical deflection is half the general relativistic deflection.

   Answer: Let have a corpuscle of light whose mass is m and whose speed at infinity (i.e. very far away from the gravitating body) is c and hence its classical total energy (i.e. kinetic plus potential) is $E = \frac{1}{2}mc^2 > 0$. Now, according to classical mechanics the orbit of this corpuscle in the gravitational field of a gravitating body is a hyperbola with the gravitating body being at one of its foci.[157] Due to the conservation of total energy (i.e. kinetic plus potential) in the conservative gravitational field the total energy of this corpuscle along its orbital path remains constant, i.e. $E = \frac{1}{2}mc^2$. Moreover, in classical mechanics the relation between the total energy E of a gravitated object of mass m that follows a hyperbolic path in the gravitational field of a gravitating object of mass M and the transverse semi-axis a of its hyperbolic path is given by:

$$E = \frac{GMm}{2a} \qquad \text{that is} \qquad a = \frac{GMm}{2E} \tag{34}$$

   Referring to Figure 4, we have:

$$\tan\left(\frac{\delta\phi}{2}\right) = \frac{a}{b} = \frac{GMm}{2Eb} = \frac{GMm}{mc^2 b} = \frac{GM}{c^2 b}$$

   where b is the conjugate semi-axis of the hyperbolic path and where in step 2 we substituted from Eq. 34 while in step 3 we used $E = \frac{1}{2}mc^2$. Now, if we consider that the deflection of light by gravity in classical systems is very tiny (i.e. $\delta\phi$ is very small) then we can use the tangent approximation for small angles in the last equation, that is:

$$\tan\left(\frac{\delta\phi}{2}\right) \simeq \frac{\delta\phi}{2} = \frac{GM}{c^2 b} \qquad \text{that is} \qquad \delta\phi \simeq \frac{2GM}{c^2 b}$$

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[157] In classical mechanics, the shape of the orbit of a gravitated body is elliptic if $E < 0$, parabolic if $E = 0$ and hyperbolic if $E > 0$.

To compare this classical formula with the general relativistic formula (i.e. $\delta\phi \simeq \frac{4GM}{c^2 d}$) we need to find the relation between b and d. Now, d in the general relativistic setting is the distance of closest approach and hence it corresponds in the classical setting to the distance between the focus (where the gravitating body is located) and the vertex of the hyperbolic branch that represents the path (see Figure 4). Accordingly, we should have:

$$\begin{aligned} d &= \sqrt{a^2+b^2}-a \qquad (35)\\ d+a &= \sqrt{a^2+b^2}\\ (d+a)^2 &= a^2+b^2\\ d^2+2da+a^2 &= a^2+b^2\\ d^2+2da &= b^2\\ b &= \sqrt{d^2+2da}\\ b &= d\sqrt{1+2\,(a/d)} \end{aligned}$$

where we used standard relations from the geometry of hyperbola (the unfamiliar reader should refer to the literature of conic sections). Now, since the deflection is very tiny (and hence the hyperbola is highly eccentric) then from the properties of hyperbola we should have $a \ll d$ and hence $(a/d) \simeq 0$ and $b \simeq d$, that is:

$$\delta\phi \simeq \frac{2GM}{c^2 d}$$

As we see, this classical deflection is half the general relativistic deflection, i.e. $\delta\phi \simeq \frac{4GM}{c^2 d}$.

Note: the above derivation can be criticized and challenged from various perspectives, e.g. the implicit assumption of the variation of the speed of light over its path in the gravitational field (because the conservation of kinetic-plus-potential energy requires the speed of light to vary along its path as the potential varies along this path). There are other classical derivations but they can also be criticized and challenged. We should also note that the approximation $b \simeq d$ may also be obtained (possibly more directly) from Eq. 35 because the tininess of the deflection means that $a \ll b$ and hence from Eq. 35 we can conclude that $d \simeq b$.

9. Since light is massless and classical gravity is a property of mass (according to Newton's gravity law), how can light be deflected by gravity in classical physics?
Answer: We note the following:
• In modern physics the light (or photon) is massless. However, in classical physics the masslessness of light is not a clear cut issue. The classical corpuscular theory of light seems to assume that light is made of massive particles (or corpuscles).
• The mass-energy equivalence according to the Poincare relation $E = mc^2$ may be derived classically (see B4) and hence it can be used to support the claim that light should have effective mass in classical physics even if its rest mass is presumed to be zero, i.e. it does not exist in a state of rest. However, the use of the relation $E = mc^2$ in this argument seems inconsistent with the derivation in exercise 8 since in that derivation we assumed $E = \frac{1}{2}mc^2$ (although the meaning of m in the two equations is not the same). Nevertheless, this potential inconsistency should not affect the essential point that is the effective mass (if accepted classically) should justify the gravitation (and hence deflection) of light classically (noting also that there are other classical derivations).

10. Calculate the deflection of a light ray grazing the Sun.
Answer: Since the ray is grazing then d in the formula of light deflection represents the Sun radius (or radial coordinate noting that the difference between the two is negligible in these classical systems) which is about 6.9551×10^8 m. Accordingly:

$$\delta\phi = \frac{4GM}{c^2 d} \simeq \frac{4 \times 6.674 \times 10^{-11} \times 1.989 \times 10^{30}}{(3 \times 10^8)^2 \times 6.9551 \times 10^8} \simeq 8.483 \times 10^{-6}\,\text{rad} \simeq 1.7497\,\text{arcsec}$$

This, of course, is the general relativistic value. The classical value is half this.

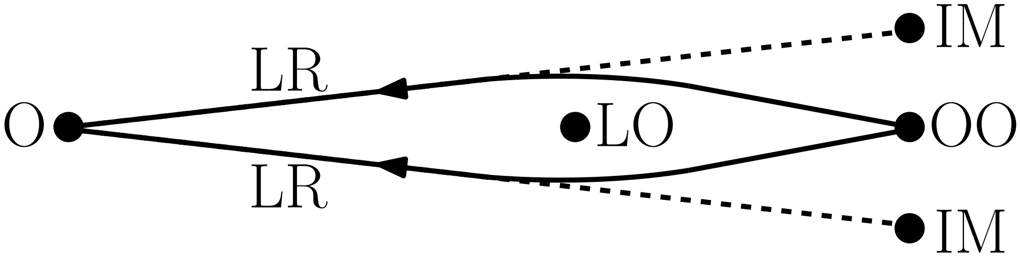

Figure 5: A simple sketch depicting gravitational lensing where the observed object OO, the observer O, the lensing object LO, the light rays LR, and the images IM in their apparent location are indicated.

11. What is the significance of the apparent displacement of the image of an observed object when the light originating from the object bends?
    Answer: This shows that our vision is "flat" (or rectilinear) in this regard unlike the supposedly-curved spacetime that we evolved in and this could put a question mark on the curvature of spacetime. However, the locality (on which our direct experiences rest) may explain the perceived flatness. Also, the smallness of the curvature (since we are supposed to be living within classical gravitational systems of weak gravity) may also explain the perceived flatness (since the accuracy of our perception system is limited).
12. Do you think "gravitational lensing" is a suitable term for the described phenomenon?
    Answer: It is a suitable term from the perspective of representing an optical phenomenon caused by gravity. However, "lensing" may be misleading because "lensing" (which is derived from lens) suggests gathering light from an object and focusing it to produce a single image (usually with positive or negative magnification) while in gravitational lensing the gathered light is not focused to produce a single magnified image of the object but multiple *images* of the object or an *image*[158] that does not depict the shape of the object such as ring. In fact, there are many formal and informal aspects by which ordinary optical lensing differs from gravitational lensing and hence the term "gravitational lensing" should be misleading. The following quote from Rindler (see References) reveals some misleading aspects: "Gravitational lenses do not bring the light to a unique focus, produce neither real nor virtual images, and, in contrast to convex glass lenses, their bending decreases with distance from the center".
13. Make a simple sketch depicting gravitational lensing.
    Answer: Refer to Figure 5.
14. What can gravitational lensing reveal about the lensing object and the lensed (or observed) object?
    Answer: Gravitational lensing may be used to judge and estimate the mass of the lensing object since the amount of light deflection is proportional to the mass of the lens according to the light deflection formula (i.e. $\delta\phi = \frac{2GM}{c^2d}$ in classical physics and $\delta\phi = \frac{4GM}{c^2d}$ in general relativity). Gravitational lensing may also be used to infer the existence and quantify the amount of dark matter (which acts as a lens) that is supposed to exist along the light path to explain the deflection.
    Regarding the observed object, gravitational lensing may reveal the existence of an object that may not be possible to observe without gravitational lensing, e.g. because of its concealment behind the lensing object. Moreover, it may be used to compute the "exact" location of an astronomical object.
    Note: "dark matter" in this answer should mean matter that is not observed directly (i.e. without the observed deflection) although there could be another independent evidence for its existence (apart from the presumed gravitational lensing) and hence this is not an admission of dark matter in the sense of modern physics (e.g. non-baryonic matter). Anyway, this could reflect one view in modern physics regardless of accepting this view or not. Also, see exercise 4 of § 3.3.
15. Show formally that the deflection of a gravitational lens decreases with the increase of the distance of closest approach of the deflected ray.
    Answer: From the light bending formulae (i.e. $\delta\phi = \frac{4GM}{c^2d}$ in general relativity and $\delta\phi = \frac{2GM}{c^2d}$ in

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[158] We note that *image* here is not in the same sense as image in optical lensing.

classical gravity) we see that the angle of deflection $\delta\phi$ is inversely proportional to the distance of closest approach d (i.e. the farthest rays are least deflected). This is unlike the bending of (convex) optical lenses where the deflection of ray is directly proportional[159] to the distance of the deflected ray from the central axis (and this is behind the focusing nature of optical lenses).

16. Show that the deflection of light by gravity can be considerable only if the gravitating object is compact (e.g. black hole).
Answer: We note first that this question is related to the previous question. The relativistic and classical light bending formulae can be written (respectively) in terms of the Schwarzschild radius R_S of the gravitating object as:

$$\delta\phi = \frac{2R_S}{d} \qquad \text{and} \qquad \delta\phi = \frac{R_S}{d}$$

Now, since $R_S \ll d$ unless the gravitating object is compact (where d can be comparable to R_S) then the deflection can be considerable only if the gravitating object is compact.

17. Show that in gravitational lensing the lensed object cannot be observed when the distance of the observer from the lens is less than a certain amount, and give an estimate of this amount.
Answer: Because the distance of closest approach d has a minimum limit (which is the radius R of the lens) the lensed object cannot be observed when the distance D of the observer from the lens is less than a certain amount. In fact, we can make a rough estimation of this amount using the tangent approximation of small angle, i.e. $\delta\phi \simeq \tan(\delta\phi) \simeq d/D$ and hence:

$$D_m \simeq \frac{d_m}{\delta\phi_x} = \frac{c^2R^2}{4GM}$$

where D_m is the minimum distance between the observer and the lens (for successful observation), d_m is the minimum limit of d and $\delta\phi_x$ is the maximum possible deflection, and where in the second step we used the general relativistic formula $\delta\phi = \frac{4GM}{c^2d}$ with $\delta\phi \equiv \delta\phi_x$ and $d \equiv d_m = R$. In fact, this estimation is based on other approximations and hence it should represent an order of magnitude estimation. However, all these approximations do not affect the main point which is the requirement of a minimum limit on the distance between the observer and the lens (for successful observation).

8.3 Gravitational Time Dilation

8.4 Gravitational Frequency Shift

1. Derive a classical formula for the gravitational frequency shift and hence compare the classical and general relativistic formulae. Also, determine how the classical formula becomes when the observer is at infinity.
Answer: The gravitational potential energy of an object O_a (say a particle) with mass m gravitated by another object O_b (say a star) with mass M is given by $\frac{-GmM}{r}$ where r is the distance between the centers of the two objects. Now, if O_a is a photon then its effective mass[160] is given by Poincare's mass-energy relation, i.e. $m = E/c^2$ where E is the photon kinetic energy[161] and c is the speed of light. So, if the distance between O_a and O_b is initially r_1 and finally r_2 then by the conservation of energy (i.e. kinetic plus potential) we have:

$$E_1 - \frac{Gm_1M}{r_1} \quad = \quad E_2 - \frac{Gm_2M}{r_2}$$

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[159] "Directly proportional" here is used in its generic meaning (without suggesting a specific mathematical correlation), i.e. the deflection increases/decreases with increasing/decreasing distance.

[160] Since we are dealing with a classical problem then m could be the mass (according to the corpuscular theory).

[161] It is also its total energy (since photons do not have rest energy) noting that total energy in this context should not include potential energy (which may be seen to belong to the field or because potential energy should not contribute to the gravitated effective mass since gravity should not act on itself; see footnote [162]).

$$\begin{aligned} h\nu_1 - \frac{Gm_1M}{r_1} &= h\nu_2 - \frac{Gm_2M}{r_2} \\ h\nu_1 - \frac{GE_1M}{c^2r_1} &= h\nu_2 - \frac{GE_2M}{c^2r_2} \\ h\nu_1 - \frac{Gh\nu_1M}{c^2r_1} &= h\nu_2 - \frac{Gh\nu_2M}{c^2r_2} \end{aligned}$$

where E_1, E_2 and m_1, m_2 are the kinetic energy and effective mass of the photon at r_1, r_2 and where in line 2 we use Planck's energy-frequency relation $E = h\nu$ (with h being Planck's constant and ν being the photon frequency) while in line 3 we use Poincare's mass-energy relation and in line 4 we use Planck's relation again.[162] Accordingly:

$$\begin{aligned} \nu_1 - \frac{G\nu_1M}{c^2r_1} &= \nu_2 - \frac{G\nu_2M}{c^2r_2} \\ \nu_1\left(1 - \frac{GM}{c^2r_1}\right) &= \nu_2\left(1 - \frac{GM}{c^2r_2}\right) \\ \nu_2 &= \nu_1\left(1 - \frac{GM}{c^2r_1}\right)\left(1 - \frac{GM}{c^2r_2}\right)^{-1} \end{aligned}$$

The general relativistic formula for the gravitational frequency shift is (see § 6.3.4):

$$\nu_2 = \nu_1\left(1 - \frac{2GM}{c^2r_1}\right)^{1/2}\left(1 - \frac{2GM}{c^2r_2}\right)^{-1/2}$$

and hence the two formulae are different. However, in most practical situations the classical and general relativistic predictions are quantitatively very close (see the upcoming exercises and refer to § 9.4).

When the observer is at infinity (i.e. $r_2 \to \infty$) the classical formula becomes:

$$\nu_2 = \nu_1\left(1 - \frac{GM}{c^2r_1}\right)$$

Note: the above derivation is not entirely classical since we use Poincare's mass-energy relation (which may not be classical; see B4) and Planck's energy-frequency relation. Hence, we can describe the derived formula as semi-classical although it is purely classical from a gravitational perspective (which is the primary perspective in our investigation here).

2. Show that the general relativistic formula for the gravitational frequency shift agrees with the classical formula (see exercise 1) by using power series approximation with reasonable conditions and assumptions.

   Answer: The general relativistic formula for the gravitational frequency shift is:

$$\nu_2 = \nu_1\left(1 - \frac{2GM}{c^2r_1}\right)^{1/2}\left(1 - \frac{2GM}{c^2r_2}\right)^{-1/2} = \nu_1\frac{\left(1 - \frac{2GM}{c^2r_1}\right)^{1/2}}{\left(1 - \frac{2GM}{c^2r_2}\right)^{1/2}}$$

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[162] In lines 3 and 4 we are assuming that the effective mass of the photon corresponds to its kinetic energy only which may be disputed. However, this can be justified by the fact that the effective mass that is influenced by gravity should not include gravitational contribution (because gravity should not act on itself). It can also be justified by claiming that potential energy should be attributed to the gravitational field rather than the gravitated object. In fact, this may even be justified (argumentatively) by being a good approximation since the kinetic energy of the photon is usually (considering the classical nature of the problem) much larger than its potential energy. Also see footnote [161].

On using power series expansion we get:[163]

$$\nu_2 = \nu_1 \frac{\left(1 + \frac{1}{2}\left[-\frac{2GM}{c^2 r_1}\right] - \frac{1}{8}\left[-\frac{2GM}{c^2 r_1}\right]^2 + \cdots\right)}{\left(1 + \frac{1}{2}\left[-\frac{2GM}{c^2 r_2}\right] - \frac{1}{8}\left[-\frac{2GM}{c^2 r_2}\right]^2 + \cdots\right)} = \nu_1 \frac{\left(1 - \frac{GM}{c^2 r_1} - \frac{G^2M^2}{2c^4 r_1^2} + \cdots\right)}{\left(1 - \frac{GM}{c^2 r_2} - \frac{G^2M^2}{2c^4 r_2^2} + \cdots\right)} \tag{36}$$

Now, if we assume that the quadratic and higher terms are negligible then we have:

$$\nu_2 \simeq \nu_1 \frac{\left(1 - \frac{GM}{c^2 r_1}\right)}{\left(1 - \frac{GM}{c^2 r_2}\right)} = \nu_1 \left(1 - \frac{GM}{c^2 r_1}\right)\left(1 - \frac{GM}{c^2 r_2}\right)^{-1}$$

which is the classical formula that we derived in exercise 1. In fact, in most cases the quadratic and higher terms are negligible and hence the power series approximation is very good. For example, in the numerator of Eq. 36 the ratio of the quadratic term $\frac{G^2M^2}{2c^4 r_1^2}$ (or rather its magnitude) to the linear term $\frac{GM}{c^2 r_1}$ at the surface of the Sun (which is a typical star) is:

$$\frac{G^2M^2}{2c^4 r_1^2} \times \frac{c^2 r_1}{GM} = \frac{GM}{2c^2 r_1} \simeq \frac{6.674 \times 10^{-11} \times 1.989 \times 10^{30}}{2 \times (3.0 \times 10^8)^2 \times 6.9551 \times 10^8} \simeq 1.060 \times 10^{-6}$$

and the ratio of the quadratic term $\frac{G^2M^2}{2c^4 r_2^2}$ to the linear term $\frac{GM}{c^2 r_2}$ in the denominator should be smaller since $r_2 > r_1$. Accordingly, the quadratic and higher terms (which are much smaller than the quadratic) in the numerator and denominator are negligible and hence the truncated power series approximation is very good.

Note: the above comparison between the classical and general relativistic formulae ignores the difference between radius (in the classical formula) and radial coordinate (in the general relativistic formula) as if they are identical (see problem 2 of § 8.3). We should also note that the triviality of the quadratic and higher terms (as measured by the ratio of the quadratic term to the linear term) may be shown more generally as follows:

$$\frac{G^2M^2}{2c^4 r^2} \times \frac{c^2 r}{GM} = \frac{GM}{2c^2 r} = \frac{1}{4}\frac{2GM}{c^2 r} = \frac{1}{4}\frac{R_S}{r}$$

where R_S is the Schwarzschild radius of the gravitating object and r is the radius (or radial coordinate). Now, for classical objects (e.g. ordinary stars such as Sun) $R_S \ll r$ and hence this ratio is very small. In fact, the condition $R_S \ll r$ should apply to all objects except black holes (and possibly neutron stars depending on the contexts and circumstances) which are exceptionally exotic objects whose existence is not definite (at least in our view).

3. Find the fractional change in the frequency of a light signal emitted on the surface of a star (say white dwarf) with one solar mass and one Earth radius when it reaches the Earth. Use in your answer both the general relativistic and classical formulae and compare the results.
   Answer: We subscript the variables on the surface of the star with 1 and the variables at the Earth with 2. Now, the gravitational field of the star at the position of the Earth is negligible and hence the Earth is at infinity. Therefore, we use the formulae at infinity.
   The general relativistic formula for the gravitational frequency shift at infinity (i.e. $r_2 \to \infty$) is:

$$\nu_2 = \nu_1 \sqrt{1 - \frac{2GM}{c^2 r_1}} \tag{37}$$

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[163] We have:

$$(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots \qquad (-1 < x < 1)$$

where ν_1 is the frequency of the emitted signal at the surface of the star, ν_2 is the frequency as observed on the Earth, G is the gravitational constant, c is the characteristic speed of light, and M and r_1 are the mass and radius (or radial coordinate) of the star and hence $M \simeq 1.989 \times 10^{30}$ kg and $r_1 \simeq 6.371 \times 10^6$ m. Accordingly, the fractional change according to general relativity is given by:

$$\begin{aligned}\frac{\nu_2 - \nu_1}{\nu_2} &= 1 - \frac{\nu_1}{\nu_2} = 1 - \left(1 - \frac{2GM}{c^2 r_1}\right)^{-1/2} \\ &\simeq 1 - \left(1 - \frac{2 \times 6.674 \times 10^{-11} \times 1.989 \times 10^{30}}{(3.0 \times 10^8)^2 \times 6.371 \times 10^6}\right)^{-1/2} \simeq -2.3159 \times 10^{-4}\end{aligned}$$

where the minus sign indicates a red shift.
Similarly, the classical formula for the gravitational frequency shift at infinity (i.e. $r_2 \to \infty$) is:

$$\nu_2 = \nu_1 \left(1 - \frac{GM}{c^2 r_1}\right)$$

Accordingly, the fractional change according to classical physics is given by:

$$\begin{aligned}\frac{\nu_2 - \nu_1}{\nu_2} &= 1 - \frac{\nu_1}{\nu_2} = 1 - \left(1 - \frac{GM}{c^2 r_1}\right)^{-1} \\ &\simeq 1 - \left(1 - \frac{6.674 \times 10^{-11} \times 1.989 \times 10^{30}}{(3.0 \times 10^8)^2 \times 6.371 \times 10^6}\right)^{-1} \simeq -2.3156 \times 10^{-4}\end{aligned}$$

As we see, the difference between the prediction of general relativity and the prediction of classical physics is negligible (despite the fact that the star is compact).

4. Write the gravitational frequency shift formula of general relativity and classical physics in terms of wavelength.
Answer: We have $\nu = c/\lambda$ where ν and λ are the frequency and wavelength. Hence, the gravitational frequency shift formula of general relativity becomes:

$$\begin{aligned}\frac{c}{\lambda_2} &= \frac{c}{\lambda_1}\left(1 - \frac{2GM}{c^2 r_1}\right)^{1/2}\left(1 - \frac{2GM}{c^2 r_2}\right)^{-1/2} \\ \lambda_2 &= \lambda_1\left(1 - \frac{2GM}{c^2 r_1}\right)^{-1/2}\left(1 - \frac{2GM}{c^2 r_2}\right)^{1/2} \qquad (38)\end{aligned}$$

Similarly, the gravitational frequency shift formula of classical physics becomes:

$$\begin{aligned}\frac{c}{\lambda_2} &= \frac{c}{\lambda_1}\left(1 - \frac{GM}{c^2 r_1}\right)\left(1 - \frac{GM}{c^2 r_2}\right)^{-1} \\ \lambda_2 &= \lambda_1\left(1 - \frac{GM}{c^2 r_1}\right)^{-1}\left(1 - \frac{GM}{c^2 r_2}\right)\end{aligned}$$

Note: the relation $\nu = c/\lambda$ may not be classical because it is based on an implicit assumption of the constancy of the speed of light (although this may be restricted to the characteristic speed rather than the observed speed).

5. Show that if $\frac{GM}{c^2 r_1} \ll 1$ then the gravitational frequency shift formula of general relativity converges to its classical counterpart at infinity (i.e. $r_2 \to \infty$).
Answer: We note first that this exercise is just an instance of exercise 2. In fact, this exercise represents the common case in astronomical observations of gravitational systems (where $\frac{GM}{c^2 r_1} \ll 1$ since the objects are usually not compact or at least not very compact and where the observations are generally made at infinity) and that is why this case deserves special attention.

For diversity, let use the gravitational frequency shift formulae that we obtained in the previous exercise in terms of wavelength (where these formulae should more appropriately be labeled as gravitational wavelength shift formulae). The general relativistic and classical formulae when the observer is at infinity (i.e. $r_2 \to \infty$) are respectively:

$$\lambda_2 = \lambda_1 \left(1 - \frac{2GM}{c^2 r_1}\right)^{-1/2} \qquad \text{and} \qquad \lambda_2 = \lambda_1 \left(1 - \frac{GM}{c^2 r_1}\right)^{-1}$$

Now, if we use series expansions (see the upcoming note) and neglect the quadratic and higher terms (since $\frac{GM}{c^2 r_1} \ll 1$) then the above formulae become:

$$\lambda_2 \simeq \lambda_1 \left(1 + \frac{GM}{c^2 r_1}\right) \qquad \text{and} \qquad \lambda_2 \simeq \lambda_1 \left(1 + \frac{GM}{c^2 r_1}\right)$$

i.e. the gravitational frequency shift formula of general relativity converges to its classical counterpart at infinity.
Note: we use the following standard power series:

$$\begin{aligned} (1-x)^{-1/2} &= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \cdots && (-1 < x < 1) \\ (1-x)^{-1} &= 1 + x + x^2 + x^3 + \cdots && (-1 < x < 1) \end{aligned}$$

We should also remind the reader that the above comparison ignores the difference between radius and radial coordinate (see problem 2 of § 8.3).

6. Use the classical and general relativistic formulations to find the wavelength of a light signal emitted at the surface of the Sun with a wavelength $\lambda_1 = 500$ nm when it reaches the Earth.
Answer: We subscript the variables on the surface of the Sun with 1 and the variables at the Earth with 2.
From the classical formula we have:

$$\begin{aligned} \lambda_2 &= \lambda_1 \left(1 - \frac{GM}{c^2 r_1}\right)^{-1} \left(1 - \frac{GM}{c^2 r_2}\right) \\ &\simeq 500 \left(1 - \frac{6.674 \times 10^{-11} \times 1.989 \times 10^{30}}{(3.0 \times 10^8)^2 \times 6.9551 \times 10^8}\right)^{-1} \times \\ &\qquad \left(1 - \frac{6.674 \times 10^{-11} \times 1.989 \times 10^{30}}{(3.0 \times 10^8)^2 \times 1.496 \times 10^{11}}\right) \\ &\simeq 500.001055412\,\text{nm} \end{aligned}$$

where r_1 is the Sun radius and r_2 is the mean distance between the Earth and the Sun.
Similarly, from the general relativistic formula we have:

$$\begin{aligned} \lambda_2 &= \lambda_1 \left(1 - \frac{2GM}{c^2 r_1}\right)^{-1/2} \left(1 - \frac{2GM}{c^2 r_2}\right)^{1/2} \\ &\simeq 500 \left(1 - \frac{2 \times 6.674 \times 10^{-11} \times 1.989 \times 10^{30}}{(3.0 \times 10^8)^2 \times 6.9551 \times 10^8}\right)^{-1/2} \times \\ &\qquad \left(1 - \frac{2 \times 6.674 \times 10^{-11} \times 1.989 \times 10^{30}}{(3.0 \times 10^8)^2 \times 1.496 \times 10^{11}}\right)^{1/2} \\ &\simeq 500.001055414\,\text{nm} \end{aligned}$$

As we see, the change in the wavelength is very tiny and beyond observational resolution. Moreover, the classical and general relativistic results are virtually identical.

Note: in this answer (as well as in the answers of similar questions) we are dealing with the problem as a purely gravitational problem and hence we are ignoring kinematical frequency shift (i.e. Doppler effect) due to the relative motion between the emitter and receiver as if the Earth and the Sun are relatively at rest. In fact, there are other causes of frequency shift which we also ignore. We should also remind the reader that in the general relativistic part we used the radius (or physical distance) for what is supposed to be radial coordinate.

7. Repeat the previous exercise but assume this time that the Sun has the same radius as the Earth.
Answer:
From the classical formula we have:

$$\begin{aligned}\lambda_2 &= \lambda_1\left(1-\frac{GM}{c^2r_1}\right)^{-1}\left(1-\frac{GM}{c^2r_2}\right)\\ &\simeq 500\left(1-\frac{6.674\times10^{-11}\times1.989\times10^{30}}{(3.0\times10^8)^2\times6.371\times10^6}\right)^{-1}\times\\ &\quad\left(1-\frac{6.674\times10^{-11}\times1.989\times10^{30}}{(3.0\times10^8)^2\times1.496\times10^{11}}\right)\\ &\simeq 500.11578\,\text{nm}\end{aligned}$$

where $r_1 = 6.371\times10^6$ m (since the Sun radius is assumed to be the same as the Earth radius).
Similarly, from the general relativistic formula we have:

$$\begin{aligned}\lambda_2 &= \lambda_1\left(1-\frac{2GM}{c^2r_1}\right)^{-1/2}\left(1-\frac{2GM}{c^2r_2}\right)^{1/2}\\ &\simeq 500\left(1-\frac{2\times6.674\times10^{-11}\times1.989\times10^{30}}{(3.0\times10^8)^2\times6.371\times10^6}\right)^{-1/2}\times\\ &\quad\left(1-\frac{2\times6.674\times10^{-11}\times1.989\times10^{30}}{(3.0\times10^8)^2\times1.496\times10^{11}}\right)^{1/2}\\ &\simeq 500.11579\,\text{nm}\end{aligned}$$

As we see, the change in the wavelength is significant and within the observational resolution. However, the classical and general relativistic results are still virtually identical and the difference between them is beyond observational resolution. This shows that even for moderately compact gravitating objects (like the one in this exercise which fits the characteristics of a white dwarf) the classical and general relativistic results are practically indistinguishable.
Again, we are ignoring in this answer kinematical frequency shift as well as other causes of frequency shift. Moreover, in the general relativistic part we use the radius (or physical distance) for what is supposed to be radial coordinate.

8.5 Gravitational Length Contraction

1. Derive a general relativistic formula for the relation between proper infinitesimal lengths along the radial direction in a Schwarzschild spacetime as obtained in two local stationary frames that are at different radial distance from the source of gravity. How this formula becomes when one frame is at infinity, i.e. very far away from the source of gravity?
Answer: If ds_1 and ds_2 are the proper infinitesimal lengths along the radial direction in frame O_1 at r_1 and frame O_2 at r_2 (where r_1 and r_2 are the radial distances[164] from a source of gravity of mass

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[164] Or rather: the radial coordinates.

M) then from Eq. 259 (in the text) we have:

$$\frac{ds_2}{ds_1} = \left(1 - \frac{2GM}{c^2 r_2}\right)^{-1/2} \left(1 - \frac{2GM}{c^2 r_1}\right)^{1/2} \tag{39}$$

Now, if O_2 is at infinity (i.e. $r_2 \to \infty$) then this formula becomes:

$$ds_\infty \equiv ds_2 = ds_1 \left(1 - \frac{2GM}{c^2 r_1}\right)^{1/2} \tag{40}$$

i.e. the proper infinitesimal length ds_∞ along the radial direction in the absence of gravity (i.e. at infinity) is shorter than the proper infinitesimal length ds_1 along the radial direction in the presence of gravity (i.e. at finite distance r_1). In other words, a stick at infinity is shorter than an identical[165] stick at finite distance from the source of gravity, i.e. the effect of gravity is to stretch sticks.

We note that the comparison of the length of two sticks at two different positions may be challenged by claiming that it is impossible or insensible or it requires the existence and adoption of a global frame which is not supposed to exist in general relativity since frames in this theory are local. However, these challenges can be revoked by the obvious possibility and sensibility of making such comparisons using calibration standards of length at infinity (or indeed at any other specific location in the spacetime). The denial of the possibility and sensibility of such comparisons should compromise the possibility and sensibility of having coordinate variables that parameterize the geometry of the spacetime. It should also affect the possibility and sensibility of gravitational time dilation because if they are denied in gravitational length contraction then they should also be denied in gravitational time dilation because both effects rest on the same physical principles.

2. What is the fractional change in the infinitesimal length along the radial direction that is experienced by a stick as it moves from the surface of the Earth to very far away from the Earth (i.e. infinity)?

Answer: Assuming a Schwarzschild spacetime, the fractional change is given by:

$$\begin{aligned}
\frac{ds_2 - ds_1}{ds_2} &= 1 - \frac{ds_1}{ds_2} \\
&= 1 - \left(1 - \frac{2GM}{c^2 r_1}\right)^{-1/2} \\
&\simeq 1 - \left(1 - \frac{2 \times 6.674 \times 10^{-11} \times 5.972 \times 10^{24}}{(3 \times 10^8)^2 \times 6.371 \times 10^6}\right)^{-1/2} \\
&\simeq -6.951 \times 10^{-10}
\end{aligned}$$

where in line 2 we use Eq. 40, and in line 3 we use the mass of the Earth for M and the radius of the Earth for r_1 (which is supposed to be radial coordinate). We note that the minus sign is consistent with the contraction of length as the distance from the source of gravity increases. We should note that strictly the fractional change due to gravitational length contraction belongs to a point on the stick since the size of length contraction depends on the radial coordinate and radial distance.

3. List some issues that can be a source of controversy and conflict in the gravitational length contraction.[166]

Answer: Examples of these issues are:

- Whether the contraction is a property of space or objects in space or both. In fact, this could be a decisive factor in determining if the contraction of objects should increase or decrease as a function of the radial distance.[167] It is relevant in this context to remember that there is a sort of conflict about the interpretation of length contraction and if it is really contraction or dilation. As indicated in the Problems, some of the conflict could be based on differences in perspective and language while some is based on employing different metrics. For example, in the Schwarzschild spacetime time and length change in opposite sense (i.e. when one dilates the other contracts and vice versa due to the reciprocal relation between the g_{00} and g_{11} coefficients of the Schwarzschild metric) but in a spacetime that is based on another metric the situation may be different and hence we may have dilation (or contraction) in both time and length or have contraction in time and dilation in length (i.e. opposite to Schwarzschild). We may add here the possibility that the origin of conflict in some cases could be the confusion about the nature of contraction and if it belongs to the objects or space or both (as indicated above).

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[165] Identical here means when they are together.

[166] This question is general and is not restricted to a specific metric like Schwarzschild.

[167] Some of the issues indicated in this point have similarities with issues related to cosmological expansion which is thoroughly investigated in the literature of relativistic cosmology.

- Whether there should be a distinction between freely falling frames and non freely falling frames and how.
- Whether the observation of gravitational length contraction is possible or not and why (and if possible how).
- Whether length contraction should enter in the analysis of the observed speed of light (like the involvement of time dilation in the analysis of Shapiro time delay; see § 9.3.2) and how.[168]
- Whether length contraction is real or apparent (as some authors seem to suggest).

We should remark that our approach in dealing with the gravitational length contraction is based on ignoring most (or all) of these controversies (or potential sources of disagreement) and treating gravitational length contraction in a similar manner to gravitational time dilation whose situation is more clear and less controversial. We should also remark that many of the conflicting views and arguments about the issue of gravitational length contraction (like many other similar issues in the relativity theories) are based on certain opinions related to the above controversies and considerations (as well as other similar controversies and considerations) and hence many of these views and arguments are incompatible with each other due to the lack of common ground.

4. In § 6.3.4 we derived gravitational frequency shift (see Eq. 251 in the text) from gravitational time dilation. Try to derive (or rather deduce) gravitational frequency shift from gravitational length contraction.

Answer: Referring to Eqs. 38 and 39, we have:

$$\frac{\lambda_2}{\lambda_1} = \left(1-\frac{2GM}{c^2r_2}\right)^{1/2}\left(1-\frac{2GM}{c^2r_1}\right)^{-1/2} \tag{41}$$

$$\frac{ds_2}{ds_1} = \left(1-\frac{2GM}{c^2r_2}\right)^{-1/2}\left(1-\frac{2GM}{c^2r_1}\right)^{1/2} \tag{42}$$

where Eq. 41 is the gravitational frequency shift formula (in terms of wavelength), while Eq. 42 is the gravitational length contraction formula. Now, if we note that ds represents a physical standard for length (e.g. of a measuring stick) while λ can be seen as a wavelength that belongs to an absolute frame (due to its relation to c) then we can conclude that contraction/dilation in ds should lead to dilation/contraction in λ and hence Eq. 41 is a natural consequence of Eq. 42, i.e. gravitational frequency shift can be seen as a result of gravitational length contraction. However, for consistency gravitational time dilation should also be taken into account.

8.6 Gravitational Waves

1. Assess the sensibility and rationality of the existence of gravitational waves.

Answer: In our view, the existence of gravitational waves is a logical aspect of gravity and hence it should be a rational and natural consequence of any modern gravity theory.[169] This can be judged from the correspondence between mass and electric charge plus the undisputed existence of

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[168] In fact, issues like the speed of light or Shapiro time delay are spacetime issues and hence a correct analysis should consider both temporal and spatial effects (as embedded in the spacetime metric).

[169] In fact, any tempo-spatial gravity theory that accepts the finity of the speed of propagation of gravitational interactions

electromagnetic waves which originate from electric charge. In fact, we believe that gravitational waves can in principle be theorized and embraced within the framework of a modern version of classical gravity that extends to include temporal aspects to this phenomenon. This view is supported by historical evidence (see the next exercise). In brief, formulating classically-based Maxwell-like gravitational equations that lead to gravitational waves and incorporating them within the framework of classical physics is quite possible in principle. However, we should note that our view does not mean the acceptance of the general relativistic interpretation of gravitational waves as ripples in the geometry of spacetime. We think gravitational waves (if they exist) are more logical to be ordinary waves like electromagnetic waves which are waves contained in spacetime and hence they are not ripples in the geometry of spacetime.
Note: the fact that the current general relativistic theory of gravitational waves is largely based on the linearized general relativity should support the possibility of establishing a Maxwell-like classical gravitational waves theory (see § 3.4).

2. Assess the commonly accepted claim that the idea of gravitational waves is the brainchild of Einstein.
   Answer: This claim is baseless. It is like many other similar baseless claims that favor Einstein regardless of the historical facts. The existence of gravitational waves was contemplated in 1893 by Heaviside in his analogy between gravitation and electrostatics which is based on the inverse square relation that governs the force in both phenomena. The historical records also indicate that in 1905 Poincare suggested (as part of his relativity theory which is based on the formalism of Lorentz transformations) the generation of gravitational waves by accelerated mass in analogy to the generation of electromagnetic waves by accelerated charge where these waves are anticipated to propagate at the speed of light. In fact, Einstein was initially critical about the existence of gravitational waves (at least as a result of general relativity and from the perspective of its formalism) although his stand was changed later. Moreover, his first attempts to formulate a general relativistic theory for gravitational waves led to non-physical solutions which were rejected even by some of his followers. At a later date he tried (with a co-worker) to publish a paper claiming the non-existence of gravitational waves within the framework of general relativity but this view was changed later when his argument was challenged and refuted.
3. Why a spherically symmetric gravitating object cannot generate gravitational waves even if it varies in time (i.e. while remaining spherically symmetric)?
   Answer: This can be justified superficially by the Birkhoff theorem (see § 4.1.1 and the Problems of the present section). However, from a more technical perspective it is shown in the literature that gravitational radiation is generated by the quadrupole moment of the mass density distribution of the gravitating object and since a spherically symmetric object has no quadrupole moment it cannot generate gravitational waves.
4. Justify the fact that gravitational waves are much weaker than electromagnetic waves (where weaker and stronger in this context may be quantified generically by the ease of detection or by the amount of energy they carry).
   Answer: It is shown in the literature that gravitational waves originate from the quadrupole moment of the mass density distribution of the source of radiation while electromagnetic waves mainly originate from the dipole moment of the charge density distribution of the source of radiation. Now, since the quadrupole moment is much weaker than the dipole moment, gravitational waves are much weaker than electromagnetic waves. We may add another reason that is the coupling of gravity to matter is weaker than the coupling of electromagnetism to charge (where being weaker may be justified by the nature of the two types of interaction).
5. Why we might have "gravity of gravity" (according to general relativity or general relativists) but not "electromagnetism of electromagnetism"?
   Answer: The source of gravity is energy-momentum while the source of electromagnetism is electric charge. Now, the gravitational field itself carries energy and hence in principle it can be a source of

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and effects should imply the existence of gravitational waves in their generic sense although the nature of these waves (such as being ripples in spacetime) should depend on the nature of the theory.

gravity (i.e. gravity of gravity)[170] but the electromagnetic field does not carry charge and hence it cannot be a source of electromagnetism (i.e. no electromagnetism of electromagnetism). In fact, this may explain (according to general relativity or general relativists) why the Field Equations of general relativity are non-linear while the Maxwell equations of electromagnetism are linear. Also see the footnote of problem 1.

6. Outline the formulation of gravitational waves in general relativity using the linearized form of the Field Equation.
   Answer: In § 3.4 we derived the following linearized form of the Field Equation:
   $$\Box^2\varepsilon_{\mu\nu} + \partial_\nu\partial_\mu\varepsilon - \partial_\nu\partial_\alpha\varepsilon_\mu^{\ \alpha} - \partial_\alpha\partial_\mu\varepsilon_\nu^{\ \alpha} - \eta_{\mu\nu}\left(\Box^2\varepsilon - \partial_\alpha\partial_\beta\varepsilon^{\beta\alpha}\right) = 2\kappa T_{\mu\nu}$$
   It can be shown that on defining $\tilde{\varepsilon}_{\mu\nu} = \varepsilon_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\varepsilon$ and imposing a certain gauge condition this equation can be written as:
   $$\Box^2\tilde{\varepsilon}_{\mu\nu} = 2\kappa T_{\mu\nu}$$
   which is a non-homogeneous wave equation (with a source term on the right hand side) that indicates the generation of gravitational waves by a source of gravity. In vacuum, $T_{\mu\nu} = 0$ and hence the latter equation becomes:
   $$\Box^2\tilde{\varepsilon}_{\mu\nu} = 0$$
   which is a homogeneous wave equation that predicts the propagation of gravitational waves in free space with speed c (noting the potential laxity of "the propagation of gravitational waves in free space").

8.7 Black Holes

1. Give a succinct and simple definition of black hole.
   Answer: It is a massive object whose mass is contained within its Schwarzschild radius (i.e. $\frac{2GM}{c^2}$ where M is its mass) and hence the escape speed at its "surface" (or event horizon) is equal to the characteristic speed of light c and therefore no light (let alone matter) can escape from its interior. In other words, matter and radiation can enter the black hole (as defined by its event horizon) but cannot leave.
2. Define event horizon and Schwarzschild radius of black hole.
   Answer:
   Event horizon of black hole is an imaginary spherical surface that is centered on the singularity (or center) of the black hole and it contains the region of space surrounding the singularity where nothing inside this region (including light) can escape to outside.
   Schwarzschild radius of black hole is the "distance" from the singularity (or center) of the black hole to the points of space where the escape speed is equal to the characteristic speed of light c and hence it is the "radius" of the event horizon.
3. In what sense black holes are "black" and "holes"?
   Answer: They are "black" in the sense that light emitted inside the event horizon[171] of the black hole cannot reach an observer outside the event horizon and hence black holes cannot be seen and observed directly by their emissions unlike other astronomical objects such as stars which can be seen and observed directly by their emissions. They are "holes" because objects (whether massive or massless) fall into them but they cannot escape from them.
4. How do we detect and observe black holes if they are "black"?
   Answer: There are several indirect ways for detecting and observing black holes. For example:
   • Black holes can be detected by their gravitational influence on neighboring objects in their surrounding, like a companion star, since their presence will be gravitationally felt by the neighboring objects whose trajectories will be modified accordingly. So, the observer can deduce their presence and infer

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[170] Being a potential source of gravity does not necessarily mean the inclusion as a source term in the Field Equations.

[171] The emission of light inside the event horizon is a hypothetical scenario whose purpose is to elucidate the fact that no light can escape from the black hole to reach the outside world.

physical information about their properties like mass from their observed influence on the neighboring objects.

- Black holes may also be detected by energetic electromagnetic emissions generated by the fall of gas, dust and cosmic debris into the black holes (i.e. before the fallen objects enter the event horizon) where these falling objects heat up as they fall into the potential well of the black hole and hence the loss in their potential energy is converted (according to the conservation of energy) to kinetic energy and consequently to radiation that can be observed by an outside observer.[172] Rotating black holes can also release energetic emissions and form jets by their rotation and magnetic fields and these emissions and jets can be detected from outside and hence their presence is deduced. Secondary mechanisms (like friction and acceleration of charged particles) can also contribute to the emission processes by converting mechanical energy to radiation.
- In theory, black holes may also be detected by emission of gravitational waves and Hawking radiation. However, these have no practical value (at least for the time being) even if we accept the existence of gravitational waves and Hawking radiation.

5. Give examples of the mechanisms by which black holes can release energy (regardless of the energy release being by the black hole on its own or through its interaction with its surrounding).

   Answer: For example:

   - By accretion of matter from the surrounding onto a (rotating or non-rotating) black hole where the lost gravitational potential energy of the accreted matter is converted to kinetic energy and electromagnetic emissions and where jets of highly energetic particles are formed. In fact, there are other secondary mechanisms of energy release based on this mechanism such as releasing energy by collision and friction between accreted particles.
   - By coupling the magnetic field of a rotating black hole to charged particles and external magnetic fields.
   - By Penrose process which is based on the fall of unstable mother particle inside the ergo-region of a rotating black hole where it decays into two daughter particles with one of the daughter particles being captured by the black hole while the other daughter particle escapes carrying more energy than the energy of the mother particle. The extracted energy according to this process is on the expense of the black hole angular momentum (which should decrease to conserve energy) and hence this process (if repeated sufficiently with no compensation of the lost angular momentum) can lead to stopping the rotating (Kerr) black hole to become a static (Schwarzschild) black hole.
   - By Hawking radiation (as will be explained in § 8.7.3). In fact, this mechanism is quantum mechanical rather than classical, i.e. it applies to a quantum (rather than a classical) black hole.[173] It also differs from other mechanisms in being energy release by the black hole itself with no interaction with conventional matter or field in its surrounding.

6. Make a rough estimate of the energy released by accretion of a certain amount of matter of mass m onto a non-rotating black hole of mass M. Comment on the result.

   Answer: The question is about a rough estimate and hence we can use the classical formulation. Also, "released" means that it can be detected outside and hence we need to find the lost potential energy (which is converted to useful energy) in moving m from infinity (i.e. where the gravitational field of the black hole is negligible) to the event horizon of the black hole (because once m is inside the event horizon no energy can be released to the outside). So, the energy released E_r is given by the magnitude of the potential energy of m at the event horizon (noting that the potential energy at infinity is zero), that is:

$$E_r = \frac{GMm}{R_S} = \frac{GMm}{(2GM)/c^2} = \frac{mc^2}{2}$$

   Now, if we note that the mass energy (according to the Poincare equation) is $E_0 = mc^2$ we see that the energy released in this accretion process is equal to half the mass energy of the accreted matter.

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[172] For simplicity and familiarity we use classical language.

[173] "Classical" in this context is opposite to "quantum mechanical" and hence even general relativity and its implications are classical in this sense.

Comment: this process of energy release (i.e. by accretion onto black hole) is much more efficient than any known nuclear reaction process (fission or fusion) since in nuclear reactions only a very tiny fraction of the mass m is converted to energy (less than 1% of the mass). In fact, the process of accretion onto black holes is the most efficient energy release mechanism if we exclude the process of annihilation of particle-antiparticle pair where the conversion is complete (i.e. 100%). We should remark that the above result represents the maximum amount of energy expected to be released from such an accretion process. In reality, the actual amount (which depends on a number of factors) is less although it is still significantly higher (even in the worst case) than the release in nuclear processes.

Note 1: some general relativistic arguments lead to the conclusion that the extracted energy is mc^2 (rather than $\frac{mc^2}{2}$).

Note 2: it may be thought that there is a fundamental difference between the energy generated by accretion and the energy generated by nuclear reaction processes where in the former case potential energy is consumed to generate other forms of energy while in the latter case mass is consumed and converted to energy. Accordingly, the mass is conserved in the former case but not in the latter case. Therefore, the above equation (i.e. $E_r = \frac{mc^2}{2}$) should not be interpreted as conversion of half of the accreted mass m into energy, but as conversion of potential energy of that quantity into other forms of energy. However, the dominant opinion in the literature of general relativity about this issue seems to be that the energy generated by accretion is the result of converting rest mass into radiation energy, and this should lead to the conclusion (among other strange conclusions) that the mass of black hole is not increased by accretion (at least not by the accreted amount but by less than this amount). In fact, some of the general relativistic arguments should lead to the conclusion that black holes (and even ordinary gravitating objects such as stars) are made of nothing. All these strange things are part of the general relativistic legacy and a result of its logic and methodology, and hence they may put question marks on the rationality of the theory (or at least some of its theoretical aspects). Our view is that from a classical standpoint the energy generated by accretion is the result of the conversion of potential energy to other forms of energy and hence the accreted mass is not affected (i.e. it is conserved), while from a general relativistic viewpoint the situation is not very clear and hence different outcomes and opinions may be claimed.[174] The source of this ambiguity in general relativity is a number of issues such as the lack of significance of potential energy and if it can contribute to mass, the lack of significance of energy (and energy conservation) on global level, and the interpretation and physical significance of binding energy. However, it seems that the most consistent view with the framework of general relativity is that the released energy comes from the accreted mass[175] (which should diminish accordingly) and hence the total mass of the accretion system (i.e. the mass of the accreted matter plus the mass of the gravitating object) diminishes by the amount of the radiated energy (and this should be consistent with the conservation of mass-energy although this conservation lacks global significance).

As a result of the above difference between classical and general relativistic views, classically the mass of the gravitating system after accretion is the sum of the mass of the system before accretion plus the mass of the accreted matter, while general-relativistically the mass of the gravitating system after accretion is this sum minus the radiated energy (which corresponds to the binding energy similar to the situation in atomic and sub-atomic systems). In fact, this may be a basis for a test that can determine which theory (if any) is the correct one from this perspective.

7. List some of the properties of black holes.

Answer: For example:

- Black holes are the most compact objects[176] in the Universe and hence they generate the strongest gravitational fields.[177] This makes them ideal candidates for testing general relativity and its implications.

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[174] In fact, this issue is a rich source of ambiguities and absurdities (which can be added to the lists of § 10.1.11 and § 10.1.20) in the theory of general relativity as can be observed from inspecting the literature.

[175] Or rather: from the accretion system.

[176] In fact, super-massive black holes have very low average density and hence if "compact" means "dense" (with average density being considered in this qualification) then this applies to certain types of black hole. Yes, if black holes are physical singularities then they are the most compact objects regardless of their type (i.e. being super-massive or not). This could also be the case if the mass of black holes can be confined within less than their Schwarzschild radius and hence their actual mass density is higher than their average mass density.

- They have very simple "structure" as they are singularities within a spacetime of simple features (unlike stars or planets for example which usually have complex structure). Moreover, they are totally characterized by only three independent physical properties: mass, angular momentum and electric charge. Accordingly, they represent matter in its most basic and degenerate form and hence they have no detailed or intricate structure such as having electrons, protons and neutrons.
- Based on the previous point, black holes are classified according to their metric as: Schwarzschild black holes (with mass only), Kerr black holes (with mass and angular momentum), Reissner-Nordstrom black holes (with mass and electric charge) and Kerr-Newman black holes (with mass, angular momentum and electric charge). This classification is comprehensive.
- They are also classified according to their mass as: super-massive, stellar and miniature. This classification is also related to their astrophysical and cosmological status and origin, i.e. being at the center of galaxies (possibly as remnants of galactic processes), being remnants of dead stars and collapsed binary systems, and being remnants of the Big Bang.
- Their "shape" is characterized by their Schwarzschild radius and event horizon. Rotating black holes also have stationary limit and ergo-region (see § 8.7.2).

8. What we mean by the structure of black hole?
   Answer: Supposedly, black holes are singularities and hence strictly speaking they do not have any structure. So, when we talk about their structure we usually mean the structure (or feature) of the spacetime that associates these black holes and surrounds them. Accordingly, black holes are just singularities in spacetime and any structure belongs to the spacetime that is shaped by them. However, as indicated earlier we do not believe that black holes are physical singularities because singularity is a mathematical, rather than physical, entity and hence it does not exist in the real world.
9. What are the main schemes for classifying black holes?
   Answer: They are:
   - Mass-based classification according to the quantity of their mass, i.e. super-massive black holes, stellar black holes, and miniature (or primordial) black holes.
   - Property-based (or metric-based) classification according to mass, angular momentum and electric charge, i.e. Schwarzschild black holes, Kerr black holes, Reissner-Nordstrom black holes and Kerr-Newman black holes.

   Note: although mass plays the primary role in the mass-based classification from a phenomenological or superficial perspective, there are more fundamental differences between these classes from the perspective of the origin of these categories of black hole and the physical processes from which they emerged.
10. Derive a mathematical expression for the Schwarzschild radius using a classical argument. What is the significance of this classical derivation?
    Answer: In classical mechanics, the escape speed v_{es} of a gravitated object of mass m in the gravitational field of a gravitating object of mass M is given by:

$$\begin{aligned} \frac{1}{2}mv_{es}^2 &= \frac{GmM}{R} \\ v_{es} &= \sqrt{\frac{2GM}{R}} \qquad (43) \end{aligned}$$

where line 1 is based on equating the kinetic energy of the gravitated object to the magnitude of its potential energy[178] (with R being the distance between the centers of the two objects). So, if this speed should be equal to the speed of light (i.e. at the Schwarzschild radius R_S) due to the fact that light cannot escape from the gravitational grip of black hole then we should have:

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[177] The equation $\nabla \cdot \mathbf{g} = -4\pi G\rho$ (associated with the divergence theorem) should give an idea about the relation between the gravitational field $\mathbf{g}$ and the mass density ρ despite its classical nature.

[178] This is because the gravitated object should have a quantity of kinetic energy that is equal to the magnitude of its potential energy (which is negative because it is trapped in the gravitational well) to be able to escape to infinity where its potential energy (as well as its kinetic energy) is zero.

$$\begin{aligned} c &= \sqrt{\frac{2GM}{R_S}} \\ c^2 &= \frac{2GM}{R_S} \\ R_S &= \frac{2GM}{c^2} \end{aligned}$$

The significance of this classical derivation is that black holes can be predicted by classical physics and easily accommodated within its theoretical framework (without the artifacts and fantasies of general relativity). This is supported by the fact that black holes (as very compact massive objects that trap even light although they are not labeled as black holes) have historical roots in the literature of classical physics where some classical physicists and scholars (including Laplace) have contemplated about the possibility of the existence of such astronomical objects prior to the emergence of general relativity, and hence black holes are neither an invention of general relativity nor an exclusive feature of this theory. In brief, black holes are not proprietary to general relativity as might be suggested by some authors. This means that any observational evidence in support of the existence of black holes cannot be regarded as endorsing evidence to general relativity against classical gravity. However, we should remark that the acceptance of black holes classically should require the acceptance of c as an ultimate speed which is not a classical result (see the upcoming note 4 and refer to the next exercise).

Note 1: although the above expression of the escape speed (i.e. $v_{es} = \sqrt{\frac{2GM}{R}}$) is derived by a classical argument, the same expression is valid in general relativity and can be obtained from the Schwarzschild metric.

Note 2: if we accept that the speed of massive objects must be less than c while the speed of massless objects is c, then at the Schwarzschild radius of black holes massless objects can escape while massive objects cannot. In other words, the region of no-escape for massive objects is defined by $r \leq R_S$ (i.e. at and inside event horizon) while the region of no-escape for massless objects is defined by $r < R_S$ (i.e. only inside event horizon). However, the requirement of having zero energy at infinity (in the definition of escape speed according to the above classical derivation) should prevent the inclusion of $r = R_S$ even for massless objects because no massless object (e.g. photon) can have zero energy.

Note 3: as explained earlier, r in the Schwarzschild metric of general relativity is a coordinate variable (rather than physical variable). On the other hand, r in any classical formulation should be a physical variable. In fact, this should highlight an important difference between what we may call "classical Schwarzschild radius" (i.e. the radius that light cannot escape from within) and the "general relativistic Schwarzschild radius". In other words, they are not the same although the formulae that represent them look misleadingly identical. Yes, if we believe that the physical radius and coordinate radius are equal (according to certain opinions as explained in problem 2 of § 8.3) then the classical "Schwarzschild" radius and the general relativistic Schwarzschild radius are the same.

Note 4: the restriction on the speed of physical objects to be less than or equal to c does not exist in classical physics and hence any purely classical derivation cannot impose a limit on the size of "Schwarzschild radius". In other words, the "Schwarzschild radius" should depend on the imposed limit on the speed of light (where this imposition is basically arbitrary unless there is a specific and legitimate physical reason for imposing such a restriction). However, the above classical derivation can still be legitimate (although it is not entirely classical) if we import the speed restriction from Lorentz mechanics, and this should not affect the classical nature (and hence the classical legitimacy of black holes) of the above derivation from a gravitational perspective although it is not classical from another perspective. Nevertheless, this may not solve the problem entirely because the speed of light is required to vanish at infinity (and should vary in between) according to the rationale of the above derivation (in violation of the constancy of the speed of light according to Lorentz mechanics). The important thing to note in this context is that the energy of radiation in classical physics is a function of its speed[179] while according to modern physics it is a function of its frequency (according to the Planck's energy-frequency relation). Yes, if we can establish a relation between the frequency and the escape speed then we may find a way for deriving the Schwarzschild radius from a classical argument (or rather semi-classical argument).

Note 5: we obtained the general relativistic expression of the Schwarzschild radius from the classical limit (see § 5.1) which is rather restricted. However, the generality of this expression may be claimed by being the requirement of the accidental singularity of the Schwarzschild metric (see the Exercises of § 4.1.1) or possibly by the above expression of the escape speed which is valid even in general relativity (although this may still be problematic). In fact, the general relativistic expression of the Schwarzschild radius may be obtained more technically from the characteristic property of the event horizon as a null hyper-surface (i.e. a surface whose normal at any point of the surface is a null vector). The details of this should be sought in the literature of black holes.

11. Assess the classical derivation of the previous exercise.

Answer: We note the following:

• The above condition (i.e. "the escape speed should be equal to the speed of light at the Schwarzschild radius" which the above derivation is based on) arises from the fact that no light should escape from inside the event horizon (whose radius is the Schwarzschild radius R_S) of a black hole because the escape of light from inside requires the escape speed to be higher than c due to the fact that the escape speed is inversely proportional to the square root of the radius, as can be seen from Eq. 43. However, for the black hole to be "black" according to this classical derivation we should accept that the speed of light cannot exceed c, and this is not a classical premise. Yes, we can take this premise from Lorentz mechanics (whose formalism is based on or leads to this premise; see B4) with no need for calling general relativity or any other gravitational theory. So, the derivation may be seen as hybrid (or semi-classical) from this perspective although it is entirely classical from a gravitational perspective (which is the important perspective in the context of comparing classical gravity to general relativity). However, as we noted in the previous exercise this may not solve the problem entirely.

• Another issue (which is rather minor) about the above derivation is that we should accept that light is gravitated (i.e. affected by gravitational force) according to classical physics. This issue has been addressed previously within the context of light bending by gravity (see exercises 3 and 9 of § 8.2).

12. What is the Schwarzschild radius of a black hole with a mass (**a**) equal to the mass of the Earth and (**b**) equal to the mass of the Sun?

Answer:

(**a**) We have $M \simeq 5.972 \times 10^{24}$ kg. Hence:

$$R_S = \frac{2GM}{c^2} \simeq \frac{2 \times 6.674 \times 10^{-11} \times 5.972 \times 10^{24}}{\left(3 \times 10^8\right)^2} \simeq 0.00886\,\text{m}$$

(**b**) We have $M \simeq 1.989 \times 10^{30}$ kg. Hence:

$$R_S = \frac{2GM}{c^2} \simeq \frac{2 \times 6.674 \times 10^{-11} \times 1.989 \times 10^{30}}{\left(3 \times 10^8\right)^2} \simeq 2949.91\,\text{m}$$

13. Use some of the previously developed general relativistic formulae related to gravitational time dilation and gravitational frequency shift to infer some of the physical properties and consequences of black holes.

Answer: For example:

• If we put $r_1 = R_S \left(= \frac{2GM}{c^2}\right)$ in the following gravitational time dilation formula (where $r_1 < r_2$):[180]

$$d\tau_1 = d\tau_2 \left(1 - \frac{2GM}{c^2 r_1}\right)^{1/2} \left(1 - \frac{2GM}{c^2 r_2}\right)^{-1/2}$$

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[179] This should be the case at least according to the corpuscular theory.

[180] This formula is a variant form of Eq. 302 (in the text).

then $d\tau_1$ becomes zero which may be interpreted as: clocks at the event horizon stop, i.e. as seen by observers outside the event horizon since $d\tau_2 > 0$ while $d\tau_1 = 0$.[181]

- If we put $r_1 = R_S$ in the following gravitational frequency shift formula (where $r_1 < r_2$):[182]

$$\nu_2 = \nu_1 \left(1 - \frac{2GM}{c^2 r_1}\right)^{1/2} \left(1 - \frac{2GM}{c^2 r_2}\right)^{-1/2}$$

then ν_2 will vanish which means infinite red shift. This can be interpreted as: no light comes from the event horizon to an outside observer because no light has zero frequency. This is inline with the Planck's energy-frequency relation $E = h\nu$ since the energy of radiation vanishes when the frequency vanishes which means there is no radiation.

14. Discuss the mathematical singularities of black holes assuming a Schwarzschild metric. Do these singularities mean that black holes are physical singularities?

Answer: The Schwarzschild metric is:

$$[g_{\mu\nu}] = \text{diag}\left[\left(1 - \frac{2GM}{c^2 r}\right), \ -\left(1 - \frac{2GM}{c^2 r}\right)^{-1}, \ -r^2, \ -r^2 \sin^2\theta\right]$$

and hence we have two singularities: one at $r = \frac{2GM}{c^2}$ (which arises from the g_{11} coefficient) and one at $r = 0$ (which arises from r being in the denominator in g_{00}). In fact, the first is an accidental (or coordinate) singularity and hence it is not really a singularity while the second (and possibly even the first) is not in the domain of the Schwarzschild solution (which is a vacuum solution) and hence we have no real singularity within the Schwarzschild metric space. Yes, the singularity at $r = 0$ can be inferred from an extended solution to include the inside of the gravitating object, as indicated earlier (see § 4.1.1). So, over the extended spacetime we have only one essential mathematical singularity.

In our view, the mathematical singularities of black holes (like the one that we found above) do not mean they are physical singularities because singularities are abstract mathematical entities that do not exist in the physical world since there is nothing infinite in the real world. In fact, because black holes can have angular momentum they should not be physical singularities. We may even claim that having mass or electric charge should also rule out the possibility of being physical singularities (since it is difficult to imagine in a sensible way that a geometric point can have mass or electric charge).

Note: historically, the singularity at $r = \frac{2GM}{c^2}$ was believed to be essential but the emergence of coordinate systems other than the Schwarzschild coordinates (e.g. Kruskal-Szekeres coordinates) changed this belief. I wonder if someone in the future can come with a coordinate system in which even the singularity at $r = 0$ disappear and hence we can get rid of all singularities. This is inline with our belief that all the claimed physical singularities are actually mathematical artifacts. In brief, if we can have a singularity that originates from the specific choice of coordinate system (which we believed to be an essential singularity until we found an alternative coordinate system) then we can similarly have an "essential" singularity that originates from the existing coordinate systems (although this "essential" singularity can disappear in a future coordinate system). In fact, we can even have an essential singularity (and hence it cannot be removed by any existing or future coordinate system) according to the existing theory (i.e. general relativity) although this essential singularity can be removed by the replacement of the existing theory with an alternative theory. So, the singularities are not physical necessities but they are mathematical artifacts that originate from the coordinate systems and physical theories and hence they could (and should) be removed by the replacement of these "defective" coordinate systems and theories with the "sound" coordinate systems and theories. This is also inline with our belief that over-mathematization of physics (which is a characteristic feature not only of the applications of general relativity and some of its details but of the theory itself since it geometrizes gravity) is a dangerous game because it leads to illusions and pitfalls since the world of mathematics is very abstract and utterly theoretical (as it should be with no criticism because this is what mathematics is supposed to be) and hence it becomes full of artifacts and illusions when it is applied literally to the reality of the physical world (as we wrongly believed in the essential nature of the singularity at $r = \frac{2GM}{c^2}$ and its consequences). In brief, mathematics (as a tool for doing physics) is a dangerous tool if it is used carelessly.

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[181] When we say "as seen by observers outside ... etc." we should mean as a limit.

[182] This formula is a variant form of Eq. 251 (in the text).

We should finally draw the attention of the readers to the existence of theorems (especially those of Penrose) that ban the removal of the "essential" singularities. However, these are essentially mathematical (rather than physical) theorems and hence they should have no physical value or consequence. Moreover, they are generally based on the theoretical framework of general relativity and hence they have very little (if any) significance from a more general perspective.

15. Can we ascribe Schwarzschild radius to objects other than black holes?

    Answer: Yes. Accordingly, the difference between black holes and other objects is that the mass of black holes is entirely confined inside their Schwarzschild radius (or rather event horizon) while the mass of other objects is not, i.e. it is partly confined inside their event horizon. So in brief, if the mass of an object becomes entirely confined (e.g. by outside pressure) inside its Schwarzschild radius then the object will become a black hole (allegedly by collapsing into a singularity); otherwise it will not be a black hole. We may also say that the radius r of an ordinary object is greater than its Schwarzschild radius (i.e. $r > R_S$) while the radius of black hole is less than or equal to the Schwarzschild radius (i.e. $r \le R_S$). In fact, if black holes are physical singularities then we should state these conditions as $r > R_S$ for ordinary objects and $r = 0$ for black holes and hence the Schwarzschild radius is not a property of the black hole itself (as a singularity) but to the surrounding spacetime although this combination of singularity and surrounding spacetime in its immediate vicinity is commonly called black hole.

    It is noteworthy that in questions like this we should assume that the object is spherical (to have a radius) or at least we define a sort of "average radius" in a sensible way. We should also remark that being "partly confined inside their event horizon" in the above definition of ordinary objects is just a possibility which is the common one; the other possibility is that no mass is confined inside.

16. What characterizes the event horizon of black holes?

    Answer: Anything (whether massive or massless) inside the event horizon cannot escape from the gravitational grip of the black hole and hence it is trapped there forever. This may be put in a rather different way by saying: the event horizon is a surface of one-way traffic. From a classical perspective, this should depend on accepting the premise that c is the ultimate speed.

17. Develop a formula for the average mass density of black holes and analyze it.

    Answer: We define the average mass density ρ_a of a black hole as the ratio of its mass M to its volume V where V is defined as the volume contained within its event horizon (as determined by its Schwarzschild radius). Hence, we have:

$$\rho_a = \frac{M}{V} = \frac{M}{(4/3)\,\pi R_S^3} = \frac{3M}{4\pi\,(2GM/c^2)^3} = \frac{3c^6}{32\pi G^3 M^2} \tag{44}$$

    As we see, the average mass density of black holes is inversely proportional to the square of mass which means that as the mass increases the average density decreases rapidly (due to the inverse square relation). In other words, the "heavy" black holes are much less dense than the "light" black holes.

    We should remark that this question is legitimate regardless of the state of black hole as being physical singularity or not because *average* density means that we are assuming the mass being distributed evenly within the event horizon regardless of being actually distributed or not. We should also remark that in the above formulation we are treating the Schwarzschild radius as an ordinary physical radius (as in Euclidean space) and this should justify the use of the ordinary formula for the volume of sphere. So, the above formulation is essentially classical.

18. Calculate the average density of a black hole whose mass is (**a**) equal to the mass of the Earth, (**b**) equal to the mass of the Sun and (**c**) equal to 10^9 solar masses. Compare these densities to the density of the atomic nucleus and comment.

    Answer: We use Eq. 44 which we developed in the previous question.

(**a**) We have $M \simeq 5.972 \times 10^{24}$ kg. Hence:

$$\rho_a = \frac{3c^6}{32\pi G^3 M^2} \simeq \frac{3 \times \left(3 \times 10^8\right)^6}{32\pi \times \left(6.674 \times 10^{-11}\right)^3 \times \left(5.972 \times 10^{24}\right)^2} \simeq 2.05 \times 10^{30}\,\text{kg/m}^3$$

(**b**) We have $M \simeq 1.989 \times 10^{30}$ kg. Hence:

$$\rho_a = \frac{3c^6}{32\pi G^3 M^2} \simeq \frac{3 \times \left(3 \times 10^8\right)^6}{32\pi \times \left(6.674 \times 10^{-11}\right)^3 \times \left(1.989 \times 10^{30}\right)^2} \simeq 1.85 \times 10^{19}\,\text{kg/m}^3$$

(**c**) We have $M \simeq 1.989 \times 10^{39}$ kg. Hence:

$$\rho_a = \frac{3c^6}{32\pi G^3 M^2} \simeq \frac{3 \times \left(3 \times 10^8\right)^6}{32\pi \times \left(6.674 \times 10^{-11}\right)^3 \times \left(1.989 \times 10^{39}\right)^2} \simeq 1.85 \times 10\,\text{kg/m}^3$$

Now, if we note that the mass density of atomic nucleus is about 2×10^{17} kg/m^3 we see that the average mass density of the black hole in part (a) is much larger than the density of atomic nucleus and the average mass density of the black hole in part (c) is much smaller than the density of atomic nucleus while the average mass density of the black hole in part (b) is almost comparable to the density of atomic nucleus. In fact, the density of the black hole in part (c) is comparable to the density of very light materials (for example the density of air under standard conditions is about 1.23 kg/m^3 and hence the density of this black hole is only about one order of magnitude higher than the density of air).

Comment: the above trend about the correlation between average density and mass can be easily explained by Eq. 44 which shows that the average density is inversely proportional to the mass squared and hence as the mass increases the average density falls sharply due to this inverse square relation. So, the results of this question demonstrate (in a quantitative form) the conclusion of the previous question, i.e. the "heavy" black holes are much less dense than the "light" black holes.

19. What can you conclude from the results of the last two questions with regard to the likelihood of the formation of black holes?

Answer: We can conclude that the likelihood of the formation of "heavy" black holes (e.g. super-massive) is higher than the likelihood of the formation of "light" black holes (e.g. miniature). This conclusion can be justified by analyzing Eq. 44 where we can see that according to this equation the mass per unit volume (i.e. mass density) that is required to make a black hole drops sharply as the mass of the black hole increases (due to the inverse square relation). Now, if we note that high mass density requires extreme physical conditions such as very high pressure (and hence it requires exceptional astronomical circumstances of rare occurrence) while low mass density does not (and hence it is very common in normal astronomical conditions) then the above conclusion should be logical.

However, we should note that the above conclusion is based on a very simple analysis and hence we expect other factors to contribute to the determination of the likelihood of the formation of black holes. Moreover, the above may be more sensible for the formation of black holes of astronomical origin (i.e. stellar and possibly super-massive) but not of cosmological origin (i.e. miniature). We should also note that the above analysis and comparison should apply even to black holes of the same type (e.g. heavy and light super-massive black holes) and not only to the different types of black hole (e.g. super-massive and miniature black holes).

20. Derive a classical formula for the gravitational field gradient of black hole along the radial direction and analyze its classical implications.

Answer: In § 1.8.2 we derived the following classical formula $\frac{dg}{dr} = -\frac{2GM}{r^3}$ for the gravitational field gradient of massive gravitating objects in general. To be specific to black holes, let express the radial distance r in terms of the Schwarzschild radius (i.e. $R_S = \frac{2GM}{c^2}$) as $r = \alpha R_S$ where α is a positive parameter. Accordingly:

$$\frac{dg}{dr} = -\frac{2GM}{\left(\alpha R_S\right)^3} = -\frac{2GM}{\left(\alpha \frac{2GM}{c^2}\right)^3} = -\frac{c^6}{4\alpha^3 G^2 M^2}$$

As we see, the gravitational field gradient (and hence the tidal force) along the radial direction is inversely proportional to the mass squared. This means that the gravitational field gradient in the vicinity of a black hole (i.e. near its event horizon) increases rapidly with the decrease of mass and therefore the gravitational field gradient is much stronger nearby a "light" black hole (e.g. miniature black hole) than nearby a "heavy" black hole (e.g. super-massive black hole).

Note: if black holes are physical singularities (which should be inconsistent with the above "classical" qualification) then α should be greater than zero (i.e. $\alpha > 0$) while if they are not physical singularities then α should be greater than some positive number $0 < \epsilon \leq 1$.

21. Considering that black holes are fully characterized by three properties: mass, angular momentum and electric charge, what are the main types of black hole from this perspective?

Answer: Since any black hole must have mass then from this perspective we have only four possibilities for the main types of black hole:

- Black hole with mass only which is a Schwarzschild black hole (i.e. the geometry of the empty spacetime around such a black hole is described by the Schwarzschild metric in the absence of other sources of gravity; see § 4.1 and § 8.7.1).
- Black hole with mass and angular momentum which is a Kerr black hole (see § 4.2 and § 8.7.2).
- Black hole with mass and electric charge which is a Reissner-Nordstrom black hole (see § 4.3).
- Black hole with mass, angular momentum and electric charge which is a Kerr-Newman black hole (see § 4.4).

22. Considering the mass range of black holes, what are the main categories of black hole from this perspective? Provide some details about these categories.

Answer: From this perspective we have three main categories:

- Super-massive black holes: the origin of these black holes is rather mysterious. In fact, there are several theories and opinions about this issue, but it is of little interest to us and hence we refer the interested reader to the literature. The mass of these black holes is in the order of millions or billions of solar masses. It is believed that most galaxies (including the Milky Way) have a super-massive black hole at their core, and this is true in particular for large galaxies (which, in the light of the conclusion of exercise 19, should be logical since the likelihood of forming such a black hole is higher). There are recent claims of definite observation of some of these black holes.
- Stellar black holes: these are created by the gravitational collapse of stars in the end of their life or by the merger of two stars of a binary system when it collapses. It is believed that stellar corpses whose mass exceeds about 3 solar masses collapse to become stellar black holes (see the upcoming note). The mass of these black holes could range between a few solar masses to tens of solar masses. There are recent claims of definite observation of some of these black holes.
- Miniature black holes: these are supposedly created during the early stages of the Big Bang due to the chaotic and violent nature of this explosion[183] which resulted in tiny spots in spacetime with extreme curvature and density, i.e. tiny black holes. The mass of these black holes is speculated to be much lower than solar mass (where estimations in the literature range between a few grams and a few multiples of the Earth mass). The existence of these black holes is based entirely on theoretical contemplations with no known claims of their observation.

Note: if the mass of stellar corpses is below the above limit (i.e. about 3 solar masses) then when the corpses collapse they become white dwarfs or neutron stars which are less compact than black holes of comparable mass. In other words, those corpses that exceed this limit are too "heavy" to become white dwarfs or neutron stars and hence when they collapse they become black holes while those corpses that are below this limit are too "light" to become black holes and hence when they collapse they become white dwarfs or neutron stars.

23. Describe briefly the free fall journey of an astronaut A toward a non-rotating black hole as experienced by him and as observed by a stationary observer B located at infinity. Assume that A is initially at infinity[184] and he is falling from rest and hence he follows a radial path toward the black hole. Also, assume that A and B have identical light sources and identical and initially-synchronized clocks.

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[183] The "chaotic and violent nature of this explosion" is rather sloppy expression.

[184] Infinity here should mean very far where the gravitational field is very small but not zero.

Answer: In this answer we present the predictions of general relativity according to the circulating literature of black holes. These predictions are generally based on the Schwarzschild metric since the black hole is non-rotating. We outline the journey and the experiences and observations of A and B in the following points:

- In the beginning there is nothing special about the journey and hence the fall as experienced by A and observed by B is like the free fall toward an ordinary gravitating object (such as a star).
- As A approaches the black hole his body starts to stretch[185] due to tidal forces and hence he can be torn apart at a certain point (depending on the mass of the black hole and the distance from its center, which determine the strength of the tidal forces according to the previous findings). However, to continue this journey and reach our destination safely we assume that A is imperishable or his protective gear includes a mechanism for resisting or annulling the tidal forces[186] (in fact we can even assume that the approached black hole is super-massive and hence its gravitational field gradient near the event horizon is not strong).
- As A approaches the black hole, it will appear to B that the light of A is red shifted and the clock of A is slowing down. On the other hand, it will appear to A that the light of B is blueshifted and the clock of B is speeding up.
- B will also observe that all light signals from A (i.e. before A reaches the event horizon) are bent except those along the radial direction.[187] In fact, not all light signals emitted by A will escape and can be observed by B. To be more clear, only those light signals confined to a certain cone (which may be called the "exit cone") that is centered on A and opens radially away from the black hole can escape the gravitational field of the black hole. Light signals outside this cone will turn back to the black hole and may not be observable by B. We should also note that light signals emitted tangentially (i.e. in a plane perpendicular to the radial direction) when A is at a distance of $1.5R_S$ from the center of the black hole will orbit (unstably) the black hole (and hence the sphere of radius $1.5R_S$ from the center of the black hole is called the photon sphere).
- When A reaches the event horizon, B will not be able to see light coming from A anymore (i.e. the light source of A is off in the eye of B) due to the infinite red shift (see exercise 13) or to the limit on the escape speed (see exercise 10). B will also observe the clock of A stopping at the event horizon (see exercise 13).[188] However, in the eye of A everything is normal, i.e. his light source is on and his clock is running normally.
- As A passes through the event horizon he will not feel anything unusual or exceptional about the event horizon. Eventually, A will be pulled to the center of the black hole where he will be squeezed into the singularity.[189]
- According to A his entire journey towards the center of the black hole takes place in a finite time, while according to B the journey of A towards the center of the black hole is never ending (see § 7.12). In fact, in the eye of B, A asymptotically approaches the event horizon in an endless journey but he will never reach it (let alone reaching the singularity).

There are many other details and variations about this journey and how it is told; however we do not need to go through these fantasies. There are also many absurd scenarios, claims and conclusions about this journey and its significance. In fact, we did our best to present this journey in a sensible and logical way.

24. State the hypothesis (or conjecture) of cosmic censorship and comment.

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[185] In fact, this does not happen in a tangible way near super-massive black holes where tidal forces near the event horizon are weak (see exercise 20).

[186] For simplicity and clarity we use here classical language as suggested for example by "tidal forces".

[187] To envisage this we need to assume that B has a sort of global presence (or at least extended presence) in spacetime so that he can observe the coming light signals from A at different locations and times.

[188] We are assuming here and in the upcoming details that a sort of communication can be established between someone outside the event horizon and someone at or inside the event horizon. We should also take note of the upcoming fact that according to B the journey of A toward the event horizon is never ending, and hence some statements here lack accuracy (but nevertheless they can be justified pedagogically and should be understood as a limit).

[189] In fact, according to general relativity any object (whether massive or massless) that crosses the event horizon (or emitted inside the event horizon) of a black hole will inevitably fall to the singularity.

Answer: The essence of this hypothesis is that gravitational singularities (as normally represented by black holes) of general relativity are hidden behind event horizons and cannot be observed from outside; the only exception (although it is not a black hole singularity) is the Big Bang which for this reason may be described as naked singularity.

Comment: this hypothesis gives the impression of a cosmic "conspiracy" against mankind to keep us ignorant (like some ancient myths of Genesis). In fact, this is an example of humanizing physics (and consequently the Universe itself) and is part of the nonsensical, subjective and illusory methodology of modern physics and cosmology. This sort of human-centered "physical" theories is common in modern cosmology and the hypothesis of cosmic censorship is used in this exercise as an example to highlight this issue. In fact, the purpose of this exercise is to exemplify a family of human-centered theorems that are common in the applications of general relativity and cosmology in particular. In our view, this type of theorems violates the spirit of physical sciences and their supposedly-objective nature (assuming cosmology is a physical science). This sort of humanization also applies (to some extent) to quantum mechanics (formalism as well as interpretation) and indeed this represents a trend in several branches of modern physics.

25. What is the implication of the "no hair theorem" (i.e. black holes are characterized by only three observable independent physical properties: mass, angular momentum and electric charge) with regard to the issue of absolute frame? What is the implication with regard to the issue of being physical singularities?

    Answer: In our view, the "no hair theorem" is an implicit confession of the absoluteness of rotation (which implies the existence of absolute frame) because angular momentum according to this theorem is regarded as an intrinsic property (like mass and electric charge) and this is not sensible unless we have an absolute frame. To be more clear, black hole can be characterized by many extrinsic properties, like position and kinetic energy, but all these properties are excluded from the "no hair theorem" because they are associated with an outside observer and hence they are extrinsic and "relative". So, if we have no absolute frame then even angular momentum should be extrinsic and relative to an observer and hence it should also be excluded from the "no hair theorem".

    Regarding the issue of being physical singularities, these properties (and angular momentum in particular) should imply that black holes are not physical singularities. The failure of black holes to be physical singularities is a failure to the theory that predicts they are physical singularities.

8.7.1 Schwarzschild Black Holes

1. Define Schwarzschild black hole in a few words.

   Answer: It is an electrically neutral and non-rotating black hole.

2. What is the photon sphere of Schwarzschild black hole?

   Answer: It is an imaginary sphere of radius $1.5R_S$ centered on the singularity of the black hole. The distinctive feature of the photon sphere is that free photons with an initial tangential velocity to the photon sphere will orbit the black hole (although the orbit is unstable).[190] Also, free photons that enter the region contained in the photon sphere from outside will inevitably be captured by the singularity although free photons emitted from within the photon sphere (but outside the event horizon) may escape the gravitational grip of the black hole and hence they might exit the photon sphere region.

3. Derive the general relativistic formula for the "radius" of the photon sphere (i.e. $r = 1.5R_S$) of Schwarzschild black hole.

   Answer: First, we refer the reader to the answer of problem 1 of § 8.2 about light bending which is based on a similar approach. The photon sphere is characterized by being the locus of the orbits of free photons circulating around the black hole and hence this problem is a problem of gravitational light bending along a circle. This problem should be investigated as a geodesic problem in the Schwarzschild metric and hence we can use the results that we obtained in § 4.1.2. Now, since light is massless then the system of geodesic equations that is suitable for investigating this problem is the system of Eqs. 164-166 (in the text). In fact, all we need from that system is Eq. 165 because the essence of this problem (i.e. a photon circulating around a black hole) is represented by a correlation between r $(= 1/u)$ and ϕ (i.e. r does not change with a change in ϕ) and that is what Eq. 165 is about. So, our investigation here starts from Eq. 165, that is:

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[190] "Free" is used here to exclude non-gravitational influences such as optical reflection by other objects. Also, "unstable" means that any incremental perturbation of r will cause the orbiting photon to spiral to infinity while any decremental perturbation of r will cause the orbiting photon to spiral into the singularity.

$$\frac{d^2u}{d\phi^2} + u = \frac{3GMu^2}{c^2} \tag{45}$$

In brief, what we need for finding a formula for the radius of the photon sphere of Schwarzschild black hole is to solve Eq. 45 to obtain the radius of the geodesic path of a photon circulating around a Schwarzschild black hole (which is the same as the radius of the photon sphere). In fact, we are assuming the existence of such a circular geodesic path which should be guaranteed by finding such a solution, i.e. the solution does exist physically because it is mathematically obtainable from the geodesic equation. We should also remind the reader that the system of Eqs. 164-166 is based on restricting the motion (and hence the geodesic) to the equatorial plane (i.e. $\theta = \pi/2$ and $d\theta = 0$) and this means that any obtained circular geodesic path is a great circle of the photon sphere (which can also be understood from the condition "circulating around a black hole").[191] This should justify the aforementioned premise that the radius of the circular path is the same as the radius of the photon sphere (and hence finding the former means finding the latter).

The condition "photon circulating around a black hole" is equivalent to the condition that the radial coordinate r of the photon does not change with a change in the azimuthal angle ϕ, i.e. $\frac{dr}{d\phi} = 0$ identically. Now, $\frac{dr}{d\phi} = 0$ implies $\frac{du}{d\phi} = 0$ (because $\frac{dr}{d\phi} = -\frac{1}{u^2}\frac{du}{d\phi}$ and u is finite) and hence $\frac{d^2u}{d\phi^2} = 0$. Therefore, Eq. 45 becomes:

$$\begin{aligned}
u &= \frac{3GMu^2}{c^2} \\
\frac{1}{r} &= \frac{3GM}{c^2r^2} \\
r &= \frac{3GM}{c^2} \\
r &= \frac{3}{2}\left(\frac{2GM}{c^2}\right) \\
r &= 1.5R_S
\end{aligned}$$

We should finally remark that photon sphere should not be proprietary to black holes since any massive object with $r < 1.5R_S$ (where R_S is the Schwarzschild radius of the object) should have a photon sphere. Apart from its obvious sensibility, this can be concluded from the above derivation where no mandatory reference to black hole is used. In fact, this can even be concluded from the Birkhoff theorem.

4. Derive a classical formula for the radius of the "photon sphere of Schwarzschild black hole" and comment on its significance.

Answer: Let assume that the photon is a massive particle (or corpuscle) circulating around a Schwarzschild black hole of mass M and hence the magnitude of its centripetal acceleration (i.e. $\frac{v^2}{r} \equiv \frac{c^2}{r}$) is equal to the magnitude of its gravitational acceleration (i.e. $\frac{GM}{r^2}$), that is:

$$\frac{c^2}{r} = \frac{GM}{r^2}$$

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[191] In other words, the orbit is contained in a plane that passes through the center of the black hole. It should be obvious that the equatorial plane (as implied by $\theta = \pi/2$) is just an instance of such a plane (due to spherical symmetry) and hence the union of all circular geodesics form the photon sphere.

$$\begin{aligned} r &= \frac{GM}{c^2} \\ r &= \frac{1}{2}\left(\frac{2GM}{c^2}\right) \\ r &= \frac{1}{2}R_S \end{aligned}$$

As we see, the radius of the "classical photon sphere" is much smaller than the radius of the general relativistic photon sphere, i.e. the latter is numerically 3 times larger than the former (but see the upcoming note 3).

Comment: if we assume that the photon sphere (as a locus of the geodesics of free photons circulating around Schwarzschild black hole) can be verified experimentally or observationally then this could be the basis of a decisive test for general relativity against classical gravity. In fact, we may not expect classical gravity to win (or even be practically verified or verifiable) but we should expect this to be a conclusive viable test for general relativity, i.e. if the radius of the photon sphere is $1.5R_S$ then general relativity is endorsed in this regard and if not then general relativity is wrong. What makes this test more conclusive than some other tests (e.g. gravitational frequency shift; see § 8.4 and § 9.4) is that the general relativistic prediction is very different from the classical prediction and hence if we assume a reasonable error margin then the results can be justified only by one theory at most (noting the possibility that both theories could fail) although the results may also be justified by a third theory.

Note 1: due to its classical nature, the question does not need the tag "Schwarzschild black hole" but we used this tag for correspondence with the general relativistic result. As we saw earlier black holes can be classical objects (i.e. any object whose mass is contained within its Schwarzschild radius or $0 < r \le R_S$ where r is the actual radius of the object) although they do not have the fantastic properties of general relativistic black holes which are obtained from the formalism of general relativity.

Note 2: as indicated earlier, the actual physical radius r of a massive object (which is more general than black hole) and its Schwarzschild radius R_S are two different things (i.e. we could have $r > R_S$ or $r = R_S$ or $r < R_S$). This is at least the case from a classical viewpoint which what we are interested in in the present question. Therefore, the obtained answer (i.e. $r = \frac{1}{2}R_S$) is physically sensible; moreover it does not need the assumption of a mathematical or physical singularity or the assumption of a point-like gravitating object. However, the mass should be contained within $\frac{1}{2}R_S$.

Note 3: the radius of classical photon sphere is metrical. So, for a fair comparison (as suggested by the aforementioned "numerically 3 times larger") between the classical and general relativistic results we may need to assume that the general relativistic radius is also metrical (due possibly to the equality of coordinate radius and metrical radius; see problem 2 of § 8.3).

Note 4: we assumed in the classical derivation that the speed of light is necessarily c. As we discussed earlier, this is not a classical assumption.

5. Investigate the singularities of Schwarzschild black hole.

Answer: We have one singularity at $r = \frac{2GM}{c^2} \equiv R_S$ which is a coordinate singularity (see § 4.1.1), and we have another singularity at $r = 0$ which is an essential singularity. However, we should remark that the latter is not a singularity of the Schwarzschild solution (although it is still a singularity of the Schwarzschild black hole) because the solution is a vacuum solution that belongs to the exterior region of the gravitating object and hence $r = 0$ is not in the domain of the Schwarzschild solution.[192] So, the singularity of black hole at $r = 0$ belongs to the solution inside the gravitating object which is not a vacuum solution.

Note: as explained earlier, essential singularity is caused by the geometry of the space (where the curvature becomes infinite at the location of the singularity) while coordinate singularity is caused by the choice of the coordinate system where the coordinate system behaves badly and hence it fails to describe the geometry of the space properly at the location of the singularity although there is no problem in the geometry of the space at that location. Hence, essential singularity cannot be removed while coordinate singularity can be removed by replacing the coordinate system.

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[192] In fact, even $r = R_S$ may not belong to the domain of Schwarzschild solution (see § 4.1.1) as it is the case in non black holes such as ordinary stars whose actual radius is larger than their Schwarzschild radius.

8.7.2 Kerr Black Holes

1. What is the implication of having rotating (or Kerr) black holes?
   Answer: In our view, one of the implications of having rotating black holes (at least theoretically) is that black holes are not physical singularities because otherwise they cannot have angular momentum, i.e. they cannot be rotating (and probably this may be indicated by the so-called "ring singularity" in the model of rotating black holes). So, being rotating should imply having finite size and hence they are not physical singularities.
   Another implication is the existence of absolute frame because angular momentum is regarded (according to the "no hair theorem") as intrinsic property (like mass and electric charge) and this cannot be justified unless we have an absolute frame (see § 8.7).
2. Why should Kerr black holes be the most common type of stellar black holes in the Universe (assuming they do exist)?
   Answer: We have only four main types of black hole (i.e. Schwarzschild, Kerr, Reissner-Nordstrom and Kerr-Newman). Now, if we exclude the last two types because it is very unlikely that electric charge will stay for long (assuming it exists initially) then we have only two possibilities: Schwarzschild and Kerr. However, since stellar black holes are formed by the collapse of stars (which are usually rotating) or by the merger of binary systems (which are necessarily rotating) then the resulting black hole should also be rotating (due to the conservation of angular momentum) and hence Kerr black holes should be the most common type of stellar black holes. In fact, even black holes that are initially static (i.e. of Schwarzschild type) can gain angular momentum by accretion of matter from their surrounding and hence they become rotating black holes (i.e. of Kerr type) and this should increase the likelihood of the existence of Kerr black holes against Schwarzschild black holes. On the opposite side, the possibility of Kerr black holes to become static (e.g. by the Penrose process assuming its reality or by other interaction and accretion mechanisms) seems less likely.
3. What is the ergo-region of a rotating black hole?
   Answer: It is the region of space between the stationary limit and the outer event horizon (as shown in Figure 8 in the text).
4. Mention some of the characteristic features of the ergo-region of a rotating black hole.
   Answer: For example:
   - Unlike objects inside the event horizon, objects in the ergo-region (i.e. inside the stationary limit but outside the event horizon) can escape from the gravitational grip of the black hole.
   - Objects in the ergo-region are dragged by the rotational motion of the black hole due to frame dragging and hence they cannot remain at rest relative to an inertial observer at infinity.
5. Give some of the characteristic features of the stationary limit of a rotating black hole.
   Answer: For example:
   - It is a surface of infinite red shift.
   - It is the locus of closest approach to the rotating black hole where the approaching object is not forced to rotate (and that is why it is called stationary or static limit).
6. Compare the event horizon and the surface of infinite red shift in the Schwarzschild and Kerr black holes.
   Answer: In the Schwarzschild black holes the two are represented by a single surface (which is the Schwarzschild event horizon), while in the Kerr black holes the two are represented by two distinct surfaces (i.e. event horizon and stationary limit which is the surface of infinite red shift) separated by the zone of ergo-region.
7. Classify the regions surrounding Kerr black holes according to the ability of a physical object (as seen by an inertial observer at infinity) to stand still and rotate in one sense or another.
   Answer: We have three main regions:
   - Outside the stationary limit: the object can rotate in both senses (i.e. with and against the rotation of the black hole) and it can also stand still.

- At the stationary limit: the object cannot rotate against the black hole but it can stand still and it can rotate with the black hole.
- Inside the stationary limit: the object is forced to rotate with the black hole and hence it cannot stand still or rotate against the black hole.

8. Try to use the terms "stationary" and "static" (as used commonly in the literature of general relativity) to describe Schwarzschild and Kerr black holes.
   Answer: As explained earlier (see exercise 11 of § 4.2), it is common in the literature of general relativity (in the context of discussing spacetime metrics) to use the term "stationary" to mean time independent and the term "static" to mean time independent plus reversible in time (where reversible in time means invariant under the transformation $dt \to -dt$ or non-rotating). We may therefore describe (rather loosely) Schwarzschild black holes as static and Kerr black holes as stationary. In fact, this should not be restricted to black holes and hence even ordinary Schwarzschild and Kerr objects (e.g. non-rotating and rotating stars) can be described as static and stationary in this sense.

8.7.3 Hawking Radiation

1. What is the essence of Hawking radiation and its basic mechanism?
   Answer: By the uncertainty principle of quantum mechanics and the vacuum fluctuation of special relativistic quantum field theory a pair of virtual particle-antiparticle can be created anywhere in the spacetime. If the pair is created near the event horizon of a black hole such that the antiparticle is inside the event horizon while the particle is outside the event horizon then the antiparticle will be trapped while the particle will escape as a real particle. The stream of the escaped particles is what is called Hawking radiation.
   Note: it is not very clear what is the proposed scenario if the particle is inside the event horizon while the antiparticle is outside the event horizon (assuming this is possible) and if it should lead to "Hawking absorption" instead of Hawking radiation. Moreover, what are the probabilities of these opposite processes and if their effects could annul each other. Some authors interpret this as emission of particles and antiparticles in equal quantities. Accordingly, the antiparticles contribute to the "positive" emission of the black hole rather than being "negative" emission or absorption. However, this interpretation seems inconsistent with the presumed change of the black hole mass in the two cases. In fact, the sensibility of Hawking radiation may require "negative" mass (where some interpretations may come to the rescue by considering the consumed mass as the energy required to convert the virtual particles to real particles). If so, then the black hole should emit equal quantities of particles and antiparticles of positive energy (or mass) and absorb the same amount of antiparticles and particles of negative energy and this should ensure the conservation of energy (although we still need to make sense of the energetically-negative particles and antiparticles).
2. Analyze Hawking radiation formula for the temperature of black hole.
   Answer: According to Eq. 304 (in the text), the temperature of a black hole solely depends on its mass M since all the other symbols in this formula are constants. Moreover, the temperature is inversely proportional to the mass and hence "light" black holes are hotter than "heavy" black holes. Accordingly, miniature black holes (if they exist) are the hottest and hence they should be the best candidates for observing Hawking radiation.
   We should also note that on differentiating Eq. 304 with respect to time t (which is the proper time of an observer at infinity) and using the chain rule we get:
   $$\frac{dT}{dt} = \frac{dT}{dM}\frac{dM}{dt} = -\frac{\hbar c^3}{8\pi k G M^2}\frac{dM}{dt} \tag{46}$$
   Now, if we assume that the mass of the black hole is not nourished by accretion from outside (or the accretion is negligible) then $\frac{dM}{dt}$ should be negative (i.e. mass is decreasing with time) and hence T increases with time. This means that as the black hole evaporates its temperature increases (while its mass decreases) rapidly in an accelerated rate.[193] Accordingly, any emitted electromagnetic radiation

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[193] The "accelerated rate" may be inferred from the inverse square dependency on M although this should require further

should gradually and rapidly shift toward high frequency which should end with gamma ray burst in the end of the black hole life. This should provide a viable test for the Hawking radiation theory by searching for the signature of such radiation emitters in astronomical observations.
Note: the above analysis is very simple and is based on treating the black hole as a black body radiator, i.e. the proposed quantum mechanical mechanism of virtual particles-antiparticles is not involved in the analysis. Otherwise, the above analysis requires further justification accordingly. However, from other results of quantum mechanics (which are not shown above) the above results are essentially correct. The details should be sought in the literature of Hawking radiation.

3. Derive a formula for the power radiated by a black hole assuming it is a black body and analyze it.
Answer: According to the Stefan-Boltzmann law, the power radiated per unit surface area of a black body is given by σT^4 where σ is the Stefan-Boltzmann constant and T is the temperature of the black body. Accordingly, the power radiated by a black hole is the product of its "surface" area $4\pi R_S^2$ (as defined by its event horizon noting possible difference between coordinate and metrical radius) times σT^4, that is:[194]

$$4\pi R_S^2 \sigma T^4 = 4\pi R_S^2 \sigma \left(\frac{\hbar c^3}{8\pi k G M}\right)^4 = 4\pi \left(\frac{2GM}{c^2}\right)^2 \sigma \left(\frac{\hbar c^3}{8\pi k G M}\right)^4 = \frac{\sigma \hbar^4 c^8}{256\pi^3 k^4 G^2 M^2} \tag{47}$$

where we used Eq. 304 (in the text) in the first equality. As we see, the power radiated by a black hole solely depends on its mass and it is inversely proportional to the mass squared and hence "light" black holes are much more powerful radiators than "heavy" black holes although the total energy output of heavy black holes over their lifetime is higher because they take longer time to evaporate passing through all temperature stages and hence they produce more energy during their lifetime. For example, if we have a black hole of 1 kg and a black hole of 2 kg then the latter will radiate the same amount of energy as the former when its mass is between 1 kg and 0 kg. However, the latter will radiate extra energy (equivalent to 1 kg) when its mass is between 2 kg and 1 kg.
Note: in the above derivation we are assuming that black holes follow Stefan-Boltzmann law which should be justified by the assumption (according to the question) of being black bodies. We should also note that the use of $4\pi R_S^2$ for the surface area may be questioned.

4. It is claimed that classical black holes violate the third law of thermodynamics. Try to justify.
Answer: Since classical black holes do not radiate, their radiation power is zero and hence by the Stefan-Boltzmann law their temperature should be zero (i.e. $T = 0$) which violates the third law of thermodynamics. In contrast, quantum black holes radiate (according to Hawking effect) and hence their temperature cannot be zero (i.e. $T > 0$), therefore they do not violate the third law of thermodynamics. However, the entire issue should depend on the applicability of the Stefan-Boltzmann law and the laws of thermodynamics to these odd (or non-conventional) physical objects.

8.8 Geodetic Effect

1. Give a brief definition of gyroscope.
Answer: It is a device whose working principle and design are based on the conservation of angular momentum (in the absence of external torques) to keep record of a fixed spatial direction while moving around in the space.

2. Find the precession angle experienced in one revolution by the spin vector of a satellite gyroscope skimming the Earth in a circular orbit at very low altitude. What is the precession angle in one year?
Answer: Because the orbit is very low we can take r to be the Earth radius (reminding the reader of the difference between radial distance and radial coordinate; see problem 2 of § 8.3). Therefore, in one revolution we have:

$$\alpha = 2\pi \left[1 - \left(1 - \frac{3GM}{c^2 r}\right)^{1/2}\right]$$

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analysis.

[194] In fact, we are treating the event horizon like a surface of an ordinary black body radiator where T is defined on it.

$$\begin{aligned}
&\simeq \quad 2\pi\left[1-\left(1-\frac{3\times 6.674\times 10^{-11}\times 5.972\times 10^{24}}{9\times 10^{16}\times 6.371\times 10^{6}}\right)^{1/2}\right]\\
&\simeq \quad 6.551\times 10^{-9}\,\text{radian}\\
&\simeq \quad 1.351\times 10^{-3}\,\text{arcsec}
\end{aligned}$$

Regarding the precession angle in one year, we can use Kepler's third law to find the periodic time T of the satellite, that is:

$$T=\left(\frac{4\pi^2 r^3}{GM}\right)^{1/2}\simeq\left(\frac{4\pi^2\times\left[6.371\times 10^{6}\right]^3}{6.674\times 10^{-11}\times 5.972\times 10^{24}}\right)^{1/2}\simeq 5061\,\text{s}$$

Accordingly, there are about 6231 revolutions per year and hence we obtain the precession angle in one year by multiplying α by 6231, that is:

$$6231\times 6.551\times 10^{-9}\,\text{radian}\simeq 2.34\times 10^{-3}\,\text{radian}\simeq 8.42\,\text{arcsec}$$

Note: the reader should note that the difference between radial distance and radial coordinate should be important here even though the gravitational field is weak and the spacetime is almost flat. This is because of the violation of locality and the accumulation of tiny relativistic effects over extended tempo-spatial interval (similar to the situation in the case of perihelion precession; see § 8.1 and § 9.1).

8.9 Frame Dragging

1. Explain frame dragging in a few simple words.
   Answer: Moving (e.g. rotating) massive objects pull the surrounding spacetime with them and hence forcing inertial frames in their vicinity to move.
2. Why is frame dragging called *inertial* frame dragging?
   Answer: It may be justified by one of the following:
   • If we consider the series of instantaneous rest frames of the dragged object along its trajectory then this effect is equivalent to dragging these inertial frames, i.e. one frame to the next.
   • It may also be seen as dragging of the frame of an object in the vicinity of the gravitating object relative to a stationary inertial frame of an observer at infinity.
   • It may also be seen as dragging of a frame which is inertial if the dragging is not happening, i.e. dragging of a frame which is otherwise an inertial frame.
3. Explain, with a simple sketch, gyroscopic precession caused by frame dragging in a satellite orbiting the Earth in a polar plane.
   Answer: Let have a gyroscope in a satellite that orbits the Earth in a polar plane where the gyroscope spin vector points in a direction parallel to the equatorial plane of the Earth. Now, assume that the gyroscope spin is initially in an axial direction, i.e. it is embedded in the orbital plane and hence it points along a line that passes through the axis of rotation of the Earth. As the Earth rotates, frame dragging will cause rotation of the orbital plane around the axis of rotation of the Earth and hence the gyroscope spin vector will deviate from its axial orientation and it ceases to be embedded in the orbital plane. This is demonstrated schematically in Figure 6.
4. What feature of the Kerr metric relates to frame dragging?
   Answer: It is the non-diagonality of the metric tensor and hence the existence of the mixed $cdt\,d\phi$ term in the quadratic form which indicates a correlation between t and ϕ. This is because non-diagonality means that the underlying coordinate system of the spacetime is not orthogonal and this is caused by the rotation which distorts the spacetime in such a way that the coordinates are forced to change from their perfect symmetry state (as in the Schwarzschild case) and cease to be orthogonal so they can describe the geometry of the distorted spacetime properly.

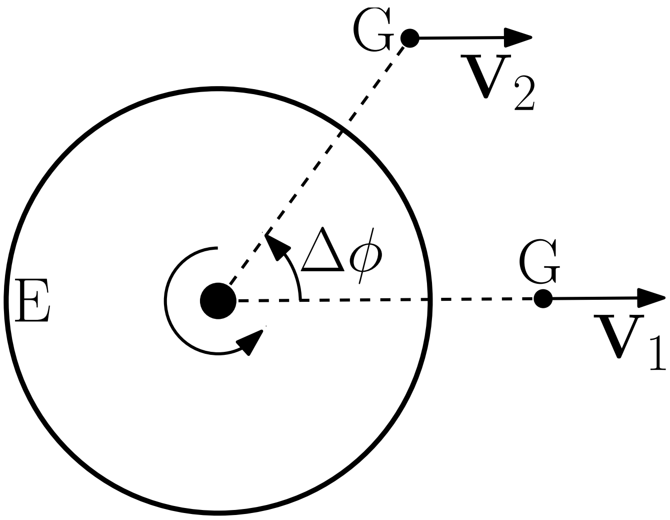

Figure 6: A polar view (i.e. along the polar axis) of the Earth E with a satellite gyroscope G whose spin vector is pointing in a direction parallel to the equatorial plane of the Earth. The gyroscope spin initially points along an axial direction $\mathbf{V}_1$ and finally points along a non-axial direction $\mathbf{V}_2$ with $\Delta\phi$ representing the angle of rotation of the orbital plane caused by frame dragging. For more clarity, we note that the equatorial plane of the Earth is represented by the plane of the page (and hence the polar axis is perpendicular to the plane of the page) while the orbital plane (which is perpendicular to the plane of the page) is represented by the dashed lines (noting that the orbital plane is polar). The sense of rotation of the Earth is indicated by the anticlockwise-rotating arrow.

8.10 Wormholes and Other Fantasies

8.11 Cosmological Predictions

Chapter 9
Tests of General Relativity

1. What is the significance of endorsing a theory by evidence that belongs to a particular aspect of the theory? What about having evidence against a particular aspect of the theory?
   Answer: Endorsing a theory by evidence that belongs to a particular aspect of the theory should endorse the theory in that particular aspect but it does not imply that the theory is right (because a wrong theory can contain correct elements or its success is fortuitous). However, any evidence against a particular aspect of the theory should lead to the collapse of the theory and the conclusion that the theory is wrong. The reason is that according to the logical convention "correct theory" means it is totally correct while "wrong theory" means it is not totally correct (i.e. it is partially or totally wrong). However, it may also be sensible to consider partial correctness and hence a theory can be partially correct and partially incorrect, e.g. classical mechanics is correct in classical systems (and hence it is still used in these systems) and incorrect in non-classical systems such as quantum and Lorentzian systems.
2. Discuss the relation between the tests of general relativity and the tests of special relativity.
   Answer: As indicated in the text, there are claims that any valid test to the theory of special relativity is a valid test to general relativity. As we will see in § 9.13, this claim can be challenged like many other similar claims. We should also note that certain tests of general relativity (especially those related to the equivalence principle and Lorentz invariance) may also be considered to be tests to special relativity. Accordingly, it may be claimed that the test of each theory should be a test to the other theory although this should be restricted to the aspects of general relativity that are directly based on the premise that special relativity holds locally in the spacetime of general relativity within freely falling non-rotating frames. In fact, there is no relation between general relativity as a gravity theory and special relativity and hence no test for one theory can be seen as a test for the other. Yes, there is a relation between general relativity as a "General Theory" and special relativity. However, the physical setting in the two theories is different (i.e. global inertial frames of flat spacetime in special relativity, and local non-inertial frames of curved spacetime which are allegedly equivalent to inertial frames in general relativity) and hence the test for each theory should belong to that theory alone and cannot be considered a test for the other theory. We may also say: the tests of general relativity as a gravity theory has no connection with special relativity because the formalisms of special and general relativity are entirely different, while the tests of special relativity cannot be tests to general relativity as a "General Theory" because the validity of the "General Theory" is based on the presumption of the validity of special relativity.
3. What correct/incorrect theory means?
   Answer: As indicated earlier, a theory is correct *iff* all its predictions are correct and incorrect *iff* some of its predictions are incorrect (where "some" may include "all" as a special case since "all" implies "some"). However, this should be regarded as a logical convention rather than a logical rule and hence we may define partial correctness/incorrectness (i.e. a theory can be partially correct/incorrect).
4. Can we have correct predictions from wrong theories, and is this consistent with the rules of logic?
   Answer: Correct predictions from wrong theories is logically consistent and historically documented where it is well known from the history of science that many wrong theories provide correct predictions and results. In fact, the progress of science (which is a long history of "mistakes") is a conclusive evidence for the fact that scientific theories generally contain both right and wrong elements (since no completely wrong theory can make progress, while no completely correct and comprehensive theory should make progress). The reality is that we do not have absolute truth and hence all our knowledge is a mix of right and wrong; therefore even the labeling of theories as "correct" and "incorrect" should be conditional and conventional. Yes, ideally we should look for and adopt a "totally correct" theory (or at least the "most correct" theory) considering pragmatic factors as well.

9.1 Perihelion Precession of Mercury

1. Is the general relativistic prediction about extra precession of orbits verified in the case of planets other than Mercury? What about other orbiting systems?
   Answer: Yes, there are such verifications although the agreement in some cases is not as good as in other cases (and in particular the case of Mercury where the agreement is "excellent"). However, the exceptionally tiny size of the extra precession of most other planets (see Table 1 in the text) should make the verification rather difficult especially when considering the involvement of other causes of precession (e.g. interaction with other planets) as well as many sources of observational error. Accordingly, this test in its alleged strong vindication to general relativity should be restricted to Mercury and some other planets and cannot be generalized to all planets.
   Regarding other orbiting systems, there are claims of highly accurate observations of periastron precession in binary orbiting systems (e.g. pulsars) that agree very well with the predictions of general relativity. If these observations have a high level of certainty and the analysis does not contain any circularity (by using general relativistic premises and results in their preparation) then this test should be more general and it could be one of the strongest evidence in support of the theory (as claimed by some).
   However, we should take into account the concerns that we discussed in the text about the validity and significance of the results of this test when assessing these perihelion/periastron observations (e.g. employing semi-classical approach and flat-space formulae in the derivation and using physical variables, in place of coordinate variables, in the calculation).

9.2 Light Bending by Gravity

1. Investigate bending of star light by the Sun and assess its value as one of the important evidence in support of general relativity.
   Answer: From a historical perspective, this is the most important single evidence that led to the general acceptance and dominance of general relativity. The evidence is based on observing the bending of star light by the Sun during solar eclipse where such observations are possible and where the results of the apparent positions of stars during the eclipse are compared to the known positions of these stars. In fact, this is seen by many as one of the strongest evidence in support of general relativity although, as we will see later, this alleged evidence is not conclusive at all even according to the opinion of some staunch supporters of general relativity.
   Bending of light rays from stars by the gravitational field of the Sun was allegedly verified by Eddington and his team in the famous solar eclipse expedition of 1919. In fact, there are some controversies about the reliability of this verification but the common opinion (which includes many supporters of general relativity) is that the analysis of Eddington and his team was biased in favor of general relativity and hence this evidence is inconclusive.[195] However, there are affirmative claims that the bending of star light by the Sun has been verified in subsequent observations where the results quantitatively agree with the predictions of general relativity within acceptable observational errors and in clear disagreement with the prediction of classical gravity. But a close inspection to these claims reveals that even later confirmations are questionable.
   Regarding the value of the claimed evidence of the 1919 expedition, as well as the value of similar later observations, we refer the reader to the paper of Ian McCausland "Anomalies in the History of Relativity"[196] where the 1919 eclipse results and conclusions are thoroughly investigated and assessed. The paper also contains discussions and quotes related to later observations of light deflection by gravity which are claimed to support the predictions of general relativity. Of particular importance is the following excerpt from this paper where McCausland quotes Sciama who refers to a table of results of eclipse observations between 1919 and 1952:

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[195] We should note that there are also technical and methodological criticisms to the 1919 expedition which further degrade the value of the findings of this expedition. The interested reader is referred to the literature for details.

[196] This paper is published in the Journal of Scientific Exploration, Vol. 13, No. 2, pp. 271-290, 1999.

It is hard to assess their significance, since other astronomers have derived different results from a re-discussion of the same material. Moreover, one might suspect that if the observers did not know what value they were "supposed" to obtain, their published results might vary over a greater range than they actually do; there are several cases in astronomy where knowing the "right" answer has led to observed results later shown to be beyond the power of the apparatus to detect. (End of quote)

In fact, the last sentence is highly eloquent and significant in highlighting a chronic problem in scientific observations where the observer sees what "should be seen" when he is driven by a theoretical model or a personal belief or other conscious or unconscious motivating factors.[197] The danger of intentional or unintentional bias in analyzing the observations is greatly magnified under the influence of Einstein and his theories where the observers normally have very strong motives to support the predictions of these theories and hence they are driven to see "what should be seen" even when they genuinely feel they are completely honest, impartial and objective.

So in brief, we can say that this "evidence" (whether from the 1919 solar eclipse observations or from later observations) is not conclusive and hence it cannot be used as evidence for endorsing general relativity despite the common claims in the literature of this theory about the value and certainty of this alleged evidence especially from the subsequent observations to those of 1919 solar eclipse. In fact, the considerable conflict in the results and evaluation of this evidence (among other factors) should degrade this evidence substantially and reduce it to be at the best of a qualitative value.

We should finally remark that there are other sources of uncertainty and error in this sort of observations due for instance to optical causes such as distortions by the optical system and refraction by the atmosphere of the Sun and the Earth. These factors (whether certain or tentative) further degrade the reliability of this type of test and make it virtually useless.

2. Discuss the historical development of the derivation of light bending formula within the framework of general relativity and the significance of this on the value of this evidence.

   Answer: Historically, bending of light by gravity was predicted and formulated (as part of the development of general relativity) using arguments based on the weak equivalence principle well before the formulation of the Field Equations. However, it is commonly claimed that the equivalence principle can quantitatively predict only half the prediction of the full general relativistic formulation. But this issue is disputed in the literature and there are claims of derivations based on the equivalence principle that agree quantitatively with the derivations that are based on the full formulation. In fact, even half the prediction could be sufficient to diminish the value of this evidence due to the uncertainty in the observations (as it is the case with the classical prediction) or at least from a qualitative perspective. So in brief, the significance of any alleged agreement between the general relativistic prediction and observations will be diminished in its endorsement to the full formulation of general relativity.

3. Examine the possible reasons for the premature endorsement of general relativity by the results of the solar eclipse of 1919.

   Answer: There are several potential reasons for this untimely endorsement and sexing up the results of this solar eclipse by Eddington and some of his colleagues in the Royal Astronomical Society. Some of these reasons (most of which can be extracted or inferred from the circulating literature of general relativity) are:[198]

   • Ideological and political motives. We should remember in this context that Eddington, like Einstein, was a pacifist and hence certain affinity and passion should have been established between the two and the effect of this psychological factor can easily extend from ideological beliefs to scientific theories. We should also remember in this regard that the endorsement of a "German theory" (i.e. general relativity) by a British team was perceived by Eddington to have beneficial healing effect on the scientific relations between Germany and Britain following the bloody conflict of the first World War.

   • Eddington's desire to make the expedition fruitful and productive. In other words, dismissing the results of the solar eclipse expedition due to the many uncertainties and errors will mean that the expedition has ended in failure and it did not achieve its objectives. So, it is logical to assume that there was a strong desire from Eddington and his team to make the results conclusive and useful (i.e. confirming the prediction of general relativity) and hence the expedition will become successful in fulfilling its goal of testing and verifying this theory.

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[197] Most people who worked in science should have first hand experience of the natural tendency to tune the parameters to match the observations when the observer has a theory and observations related to that theory (where this tendency is unconsciously motivated by the anticipated joy of success or at least by curiosity and inquisitiveness).

[198] Again, we refer the reader to the paper of McCausland "Anomalies in the History of Relativity" for more details.

- Personal belief in the correctness of this theory. This is particularly true in the case of Eddington whose faith in the theory was unshaken regardless of any evidence.
- Personal gain from supporting this theory because the work and name of Eddington and his team will be permanently attached to this theory and the history of science as it did (although sometimes for wrong reasons).

Note: most of the above reasons (as well as other similar reasons) apply not only to Eddington and his team in their premature endorsement to general relativity by the questionable results of the 1919 solar eclipse expedition, but they similarly apply to many other (old and contemporary) scientists and scholars in their enthusiastic support and endorsement to general relativity by the questionable results of their projects and investigations.

4. Briefly discuss and assess gravitational lensing as one of the claimed evidence in support of general relativity.

   Answer: It is reported that gravitational lensing, in the form of Einstein cross and Einstein ring (see § 8.2), has been observed in a number of astronomical investigations. However, we note that in some cases (at least) the observations that are supposedly based on gravitational lensing may be caused by an optical rather than a gravitational phenomenon. Moreover, from a quantitative perspective the results in most cases are highly uncertain due to the many sources of uncertainty in such astronomical gravitational systems and their observations. So, even if we assume that the results are qualitatively justifiable (i.e. they are caused by gravity) we need more justification for the quantitative aspect of these results. So in brief, this alleged evidence has at the best only qualitative endorsing value. As we know, gravitational light bending (which gravitational lensing is based on) can be explained from a pure qualitative perspective even by classical gravity (as well as by other non-classical gravity theories). In fact, it can be explained even from a quantitative perspective by other theories (including possibly classical gravity) since there are non-relativistic predictions that agree quantitatively with the relativistic predictions and hence the evidence cannot endorse general relativity specifically. We should also remark that convincing explanation should be provided for the formation of Einstein cross because the symmetry requires the formation of a ring rather than 4 images and any deviation from the symmetry should be reflected in the distortion of the ring rather than the formation of 4 separate and distinct images that are almost symmetrically distributed around the lensing object. The given explanations to the formation of a cross by the alleged gravitational lensing (e.g. some images are from direct viewing while others are from bending) are generally less convincing than optical explanations because optical causes can be more selective and random than gravitational causes.

5. Talk about the tests of light bending by gravity in the radio band (rather than the optical band) of the electromagnetic spectrum.

   Answer: These tests use VLBI technique (see the upcoming note) to detect signals from astronomical objects (like quasars) that emit electromagnetic radiations in the radio band of the spectrum. It is claimed in the literature that observations of light bending by gravity in the radio band from a number of astronomical radio sources confirm the general relativistic prediction to high level of accuracy.

   Note: VLBI is an acronym for Very Long Baseline Interferometry which is a technique used in radio astronomy. The technique is simply based on using several radio telescopes in different locations on the Earth to detect radio signals from an astronomical object simultaneously[199] where these signals are combined and used to reconstruct an "image" of the astronomical object. In essence, the multiple radio telescopes work in unison as a single giant telescope and hence the resolution of the reconstructed image improves significantly and is superior to the resolution achieved by using a single telescope.

6. Investigate the sources of error and uncertainty in the results of astronomical and cosmological investigations in general.

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[199] Simultaneous here should mean something like concurrent or synchronized or coexisting although the detections may not be exactly at the same time.

Answer: Apart from the standard sources of error and uncertainty in scientific investigations in general (such as uncertainties in the physical constants, limits on resolution, computational approximations, etc.) almost all astronomical and cosmological investigations are based on highly hypothetical physical and mathematical models with many tentative assumptions and conjunctures as well as gross theoretical approximations. A minimum familiarity with these subjects should be convincing that the results of most investigations in these fields have large margin of error and uncertainty. The reason is that the subject of these investigations are physical systems that are very far in space and time and hence they are generally not accessible to direct inspection and examination. Therefore, large amount of guesswork should be involved in modeling these systems and analyzing the data which are usually highly contaminated with noise. Moreover, the subject of these investigations is physical systems that are highly complex and involve many agents, processes, factors, etc. and hence their analysis is very messy and uncertain; furthermore any realistic analysis should be based on many simplifications and compromises which should reduce the confidence in the results. In fact, it may even be claimed that some of these systems are unlike the physical systems that we are familiar with and hence we do not have the required evolutionary experiences to fully understand these systems. This is similar to our situation in dealing with quantum mechanical systems which behave (in our view) in a strange and perplexing way because our perception system has evolved from experiences and observations on a macroscopic scale and hence our classical physics (whose roots originate from these macroscopic experiences) fails to deal with these quantum systems.

9.3 Gravitational Time Dilation

1. What are the cases in which gravitational time dilation cannot be explained classically?
   Answer: We may mention two main cases:
   • When the classical formula for gravitational frequency shift does not apply due to violation of the assumptions and conditions which the derivation of the formula is based on. This is particularly true in extreme gravitational systems where classical gravity is not expected to apply.[200] In fact, these systems should provide the decisive evidence in support and against the formulation of time dilation and frequency shift of the two theories (i.e. classical gravity and general relativity).
   • When the timing mechanism does not involve frequency. However, almost all the known timing mechanisms involve frequency. We should notice in this context that the classical formula (as derived in § 8.4) uses a photon model as a gravitated object and hence the frequency (and thus the timing mechanism) may be restricted for this reason to electromagnetic radiation. However, these issues are not clear cut and hence they can be subject to reflections and debates.
   Note: we may need to add another case that is when gravitational time dilation is not based on gravitational frequency shift (as explained in the footnote in the main text).

9.3.1 Hafele-Keating Experiment

1. Make a list of potential criticisms to the Hafele-Keating experiment and its analysis.
   Answer: Some of the potential criticisms are:
   • The experimental setting is too complicated to lend itself to a simple analysis and hence any result will not be conclusive due to the involvement of many sources of uncertainty and error. For example, the experiment involves both special and general relativistic effects (i.e. kinematical and gravitational time dilation) and hence the exact contribution of each theory to the total time dilation cannot be determined with certainty from the experimental results to compare with the theoretical predictions.[201] There are also other factors that should be taken in the analysis such as the variation in the speed,

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[200] Also, in extreme gravitational systems the power series approximation should lose its validity and hence the classical and general relativistic formulations are expected to differ significantly (regardless of the validity and invalidity of the classical gravity).

[201] In fact, the Hafele-Keating test is similar to the upcoming global positioning system test (see § 9.3.4) in the involvement of both relativity theories and hence they share the main reservations and criticisms from this perspective although their analysis may differ in some details.

path and height of the planes during the take-off and landing as well as during the flight and all these factors complicate the analysis and increase the uncertainty in the results.

• The analysis of the results requires the validity of superposition which is not obvious.

• There is a possibility of contamination of the collected data by errors from a number of causes (e.g. mechanical vibration and electric or magnetic fields). In fact, even if the atomic process on which the functionality of the cesium clocks is based was stable and cannot be affected by such causes the electronics of the clocks can be affected by these causes. Although special containers were used (according to the reports) to protect the clocks from exterior influences and sources of noise, it is difficult to believe that these containers can eliminate all these causes.

• The analysis seems to employ an external frame (i.e. the frame of the center of the Earth; see the upcoming note) relative to which the ground frame (i.e. where the stationary reference clocks on the Earth remained) is moving because without this employment the difference in sign between the kinematical time difference in the eastward and westward flights (i.e. the clocks run faster in one direction and slower in the opposite direction) cannot be justified. In fact, the employment of such a frame is similar or equivalent to having an absolute frame. In brief, the kinematical time dilation of the traveling clocks should take place with respect to the frame of the stationary reference clocks and not with respect to any other frame if we have to follow the commonly recognized interpretation of special relativity. However, following this interpretation will not solve the problem because according to the first postulate of special relativity (i.e. the relativity principle) time dilation should take place in each frame relative to the other frame and this will lead us to the complications and controversies of the twin paradox with no justification for some rebuts of the twin paradox such as "traveling twin is distinguished by being non-inertial" because both the stationary and moving frames are non-inertial or "calling for general relativity" because supposedly general relativity is already in use (refer to B4). We should also note that the introduction of this external frame introduces other complications and this should complicate the analysis further and introduce more sources of uncertainty onto the results.

• The Earth is not an inertial frame even if we consider that it is in a state of gravitational free fall (and hence it is locally inertial) because the Earth (on its surface) is a rotating frame. Yes, the center of the Earth may be considered locally inertial and that is what they seem to assume. However, although the center of the Earth may be considered locally inertial the analysis should consider potential errors that may be significant in this setting (due mainly to violation of locality as we will see). In fact, none of the frames involved in this experiment (i.e. frame of planes, frame of stationary reference clocks and frame of center of the Earth) is really inertial or can be treated as inertial. We note that although the latter frame should be equivalent to a local inertial frame (according to the equivalence principle since it is in a state of free fall) this frame is not inertial over the spacetime interval of the journey due to the violation of the locality condition over this extended interval.

• Gravitational time dilation in this experiment may be explained classically as gravitational frequency shift due to the correlation between gravitational time dilation and gravitational frequency shift (although this may be challenged because there is no actual frequency shift in this setting; see § 9.3). Moreover, it may be explained by other modern gravity theories and even by the equivalence principle without need for the formalism of general relativity. Hence, general relativity is not a necessity for the explanation of the results of this experiment in its gravitational side and therefore the experiment is not definite in endorsing general relativity.

Note: in their paper titled "Around-the-World Atomic Clocks: Predicted Relativistic Time Gains" Hafele and Keating state: "Because the earth rotates, standard clocks distributed at rest on the surface are not suitable in this case as candidates for coordinate clocks of an inertial space. Nevertheless, the relative timekeeping behavior of terrestrial clocks can be evaluated by reference to hypothetical coordinate clocks of an underlying nonrotating (inertial) space". Then in a footnote they state: "It is important to emphasize that special relativity purports to describe certain physical phenomena only relative to (or from the point of view of) inertial reference systems, and the speed of a clock relative to one of these systems determines its timekeeping behavior [G. Builder, Aust. J. Phys. 11, 279 (1958)]. Although inertial systems are highly specialized, they have an objective physical relationship with the universe because they have no acceleration or rotation relative to the universe". So, they seem to suggest the identification or association of the frame of the Earth (as represented by its center) with an absolute "inertial space".

2. Discuss briefly the "general relativistic version" of the twin paradox.
Answer: In fact, this version is the same as the special relativistic version. The only possible difference is in addressing this paradox where some authors claimed that this paradox can be addressed by general relativity specifically since it involves accelerating frames. According to these claims (which follow the theoretical framework of general relativity), the stay at home twin follows a geodesic world line (since he is not accelerating) and hence he is the older because geodesic trajectories maximize proper time, while the traveling twin follows a non-geodesic world line (since he is accelerating) and hence he is the younger. However, this alleged fix fails to address any (or at least most) of the challenges that we discussed in B4 (which the reader is referred to for details). For example, most of the aging takes place while the traveling twin is also inertial; moreover in the absence of absolute frame acceleration is relative and hence each twin (as well as both) can be accelerating.

9.3.2 Shapiro Time Delay Test

1. Give two formulae that allegedly represent Shapiro time delay and compare their quantitative predictions.
Answer: We quote and use the following two formulae which we found in the literature:

$$\Delta T = \frac{4GM}{c^3}\left[\ln\left(\frac{4d_e d_p}{R^2}\right)+1\right] \tag{48}$$

$$\Delta T = \frac{4GM}{c^3}\left[\ln\left(\frac{4d_e d_p}{R^2}\right)\right] \tag{49}$$

where ΔT is the Shapiro time delay in the round trip (i.e. onward and return), M is the mass of the Sun, d_e is the radial distance between the Earth and the Sun, d_p is the radial distance between the planet[202] and the Sun, and R is the radius of the Sun. We note that there are other formulae that we found in the literature, e.g. $\Delta T = \frac{4GM}{c^3}\left[\ln\left(\frac{d_e d_p}{R^2}\right)+1\right]$ or $\Delta T = \frac{4GM}{c^3}\left[\ln\left(\frac{4d_e d_p}{R^2}\right)-1\right]$. Although some of these formulae might be misprint, they still represent a source of confusion and uncertainty as well as a potential cause for error in the formulation and calculation. We should also note that there are differences in symbols and their interpretation (e.g. difference in using coordinate or metrical variables). However, we use our set of unified symbols and refer the reader to the literature for any potential difference or conflict.

Regarding the comparison between the quantitative predictions of these formulae (i.e. Eqs. 48 and 49), let take Mercury and Venus as reflectors and use their mean distances from the Sun, as well as the mean distance between the Earth and the Sun, for the purpose of estimation.[203] We present our comparison results in Table 1. As we see, the percentage difference between the predictions of the two formulae is significant in both cases. Accordingly, we can conclude that the uncertainty in the theoretical foundation and pertinent formulation (combined with the allowed approximations and unavoidable errors) of this time delay casts a shadow over the reliability (or at least the conclusivity) of this test because we do not know (considering the theoretical and practical uncertainties and controversies) which of these methods actually represents general relativity and is consistent with its framework and with other established facts and if it is the one that allegedly matches the experimental results very well. Hence, any claim of reliable verification of general relativity by this test should be based on a pick and choose approach. As indicated above, there are other formulae in the literature for the Shapiro time delay, and some of these formulae may produce even larger percentage difference (thus making the pick and choose approach even easier and the confidence in this test and its value even lower).

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[202] We mean the planet that reflects the signal such as Mercury and Venus. The reflector can also be an artificial satellite. We should note that "radial distance" and "radius" in this context are provisional as discussed earlier (in fact this can be a significant source of uncertainty and error in the analysis and calculation as indicated in the text).

[203] The use of the mean distances should not affect the validity of our comparison. Also, the validity of the comparison should not be affected even if we take into account the difference between coordinate variables and physical variables.

Table 1: Comparison between the quantitative predictions of two Shapiro time delay formulae using $d_e = 149.6 \times 10^9$ m and $R = 6.9551 \times 10^8$ m.

	Mean distance d_p	Formula 1 (Eq. 48)	Formula 2 (Eq. 49)	Difference
Mercury	57.9×10^9 m	$\Delta T \simeq 0.2395$ ms	$\Delta T \simeq 0.2199$ ms	8.21 %
Venus	108.2×10^9 m	$\Delta T \simeq 0.2518$ ms	$\Delta T \simeq 0.2321$ ms	7.81 %

2. Give some examples of expected sources of uncertainty and error in the Shapiro time delay test and assess their effect.

   Answer: There are "optical" sources of uncertainty and error in this test such as the refraction of the signal by the solar corona. The difference between coordinate variables and physical variables should be another source of uncertainty and error in the results if one is used in place of the other (as it seems inevitable due to the previously-discussed practical difficulties). A third example is the lack of clarity about the theoretical foundation and formulation of this test. In fact, these sources of uncertainty and error (as well as other theoretical and practical sources) should make this test and its alleged results virtually useless.

   It is worth noting that the high level of accuracy in the results of this test (as reported in the literature) requires very high level of accuracy in the measurement of distances. In fact, some of the reported accuracy values require the determination of the involved distances to an accuracy of the order of one kilometer which is unrealistic in these astronomical tests and in such dynamical orbiting systems (where distances vary continuously in time).

9.3.3 Gravity Probe A

1. What is the fractional change in the time interval between a clock on the surface of the Earth and a clock at an altitude of 10^4 km above the surface of the Earth (i.e. as in the Gravity Probe A experiment)?[204]

   Answer: We subscript the variables on the surface of the Earth with 1 and the variables at 10^4 km above the surface of the Earth with 2. The fractional change is given by:

$$\begin{aligned}
\frac{d\tau_2 - d\tau_1}{d\tau_2} &= 1 - \frac{d\tau_1}{d\tau_2} \\
&= 1 - \left(1 - \frac{2GM}{c^2 r_1}\right)^{1/2} \left(1 - \frac{2GM}{c^2 r_2}\right)^{-1/2} \\
&\simeq 1 - \left(1 - \frac{2 \times 6.674 \times 10^{-11} \times 5.972 \times 10^{24}}{\left(3 \times 10^8\right)^2 \times 6.371 \times 10^6}\right)^{1/2} \times \\
&\quad \left(1 - \frac{2 \times 6.674 \times 10^{-11} \times 5.972 \times 10^{24}}{\left(3 \times 10^8\right)^2 \times \left(10^7 + 6.371 \times 10^6\right)}\right)^{-1/2} \\
&\simeq 4.246 \times 10^{-10}
\end{aligned}$$

   where in line 2 we use Eq. 245 (in the text), in line 3 we use the mass of the Earth for M and the radius of the Earth for r_1, and in line 4 we use the mass of the Earth for M and $10^7 + r_1$ for the distance from the center of the Earth to the height of 10^4 km ($= 10^7$ m) above the surface of the Earth. We should draw the attention to the problem of using coordinate or metrical variables since r_1 and r_2 are supposed to be non-metrical coordinate variables (see § 6.2 and problem 2 of § 8.3).

   Note: we did not derive a classical formula for the gravitational time dilation because there is no time dilation in classical physics. However, in § 8.4 we derived a classical formula for the gravitational frequency shift using the conservation of energy. Now, since the Gravity Probe A experiment is based on using the frequency of maser for time measurement, we can use our classical formula to compare (indirectly) between the prediction of general relativity and the prediction of classical gravity. As we will see in § 9.4.3, the comparison between the classical and general relativistic predictions reveals that this experiment is not conclusive because the two predictions are virtually identical. Hence, the claim that the Gravity Probe A experiment endorses the general relativistic prediction of gravitational time dilation can be questioned since the result is inconclusive in its endorsement to general relativity specifically because it can be explained classically. However, the classical explanation may be challenged because the conservation of energy may apply only when there is an actual frequency shift as in the case of a signal ascending or descending in a gravitational potential well, and this is not the case in the Gravity Probe A experiment because there is no ascent or descent of a signal but instead there is a generation of two independent signals at two potential levels. Anyway, gravitational time dilation in the Gravity Probe A test can also be explained by other non-classical gravity theories and possibly even by the equivalence principle alone.

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[204] The question is restricted to the change caused by gravitational time dilation due to difference in height (or potential level) and hence it does not include other possible causes.

9.3.4 Global Positioning System

1. Calculate the general relativistic time correction that is required to keep the GPS functioning.[205]
Answer: The satellites of the GPS are at about 20200 km above the surface of the Earth. So, if we subscript the variables on the surface of the Earth with 1 and the variables at 20200 km above the surface of the Earth with 2 then we have:

$$\begin{aligned}
\frac{d\tau_2}{d\tau_1} &= \left(1-\frac{2GM}{c^2 r_2}\right)^{1/2}\left(1-\frac{2GM}{c^2 r_1}\right)^{-1/2} \\
&\simeq \left(1-\frac{2GM}{c^2\left(r_1+2.02\times 10^7\right)}\right)^{1/2}\left(1-\frac{2GM}{c^2 r_1}\right)^{-1/2} \\
&\simeq \left(1-\frac{2\times 6.674\times 10^{-11}\times 5.972\times 10^{24}}{\left(3\times 10^8\right)^2\times\left(6.371\times 10^6+2.02\times 10^7\right)}\right)^{1/2}\times \\
&\quad \left(1-\frac{2\times 6.674\times 10^{-11}\times 5.972\times 10^{24}}{\left(3\times 10^8\right)^2\times 6.371\times 10^6}\right)^{-1/2} \\
&\simeq 1.00000000052844
\end{aligned}$$

where in line 1 we use Eq. 245 (in the text), in line 2 we add 20200 km ($= 2.02\times 10^7$ m) to the Earth radius r_1 to obtain r_2, and in lines 3 and 4 we use the mass of the Earth for M and the radius of the Earth for r_1 (noting that we are using metrical distances). So, when a clock on the Earth runs $24\times 3600 = 86400$ s (i.e. one day) an identical clock at 20200 km above the surface of the Earth will run:

$$1.00000000052844\times 24\times 3600 \simeq 86400.000045658\,\text{s}$$

This means that the clock at 20200 km above the surface of the Earth is ahead of the clock on the surface of the Earth by about 45.7 μs per day. In terms of distance (using c as a conversion factor) this time difference is equivalent to:

$$0.000045658\times 3\times 10^8 \simeq 13697\,\text{m per day}$$

which is significant.
Note: the above formulation and calculation are based on the Schwarzschild metric. However, Kerr metric may be used instead since the Earth is rotating. Although this may not make a significant difference it should be desirable to ensure higher accuracy in such highly sensitive system.

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[205] In fact, the question is about the main general relativistic correction caused by gravitational time dilation due to difference in height or potential level (noting that other general relativistic corrections are required).

2. Calculate the special relativistic time correction that is required to keep the GPS functioning. Comment on the method of calculation.
Answer:[206] Assuming that the satellites in the GPS execute rotational motion in a stable circular orbit, the magnitude of their centripetal acceleration $\frac{v^2}{r}$ should be equal to the magnitude of their gravitational acceleration $\frac{GM}{r^2}$, that is:

$$\begin{aligned} \frac{v^2}{r} &= \frac{GM}{r^2} \\ v^2 &= \frac{GM}{r} \end{aligned}$$

where v is the orbital speed of the satellite, M is the mass of the Earth and r is the radius of the satellite orbit (i.e. distance from satellite to Earth center).[207] Now, according to special relativity a clock on the satellite will be observed from the frame of the Earth to run slower due to kinematical time dilation where the time interval Δt_1 on the surface of the Earth is related to the time interval Δt_2 at 20200 km above the surface of the Earth by the kinematical time dilation formula, that is:

$$\begin{aligned} \Delta t_1 &= \frac{\Delta t_2}{\sqrt{1-(v^2/c^2)}} \\ \frac{\Delta t_2}{\Delta t_1} &= \sqrt{1-(v^2/c^2)} \\ \frac{\Delta t_2}{\Delta t_1} &= \sqrt{1-\frac{GM}{rc^2}} \\ \frac{\Delta t_2}{\Delta t_1} &\simeq \sqrt{1-\frac{6.674\times10^{-11}\times5.972\times10^{24}}{(2.02\times10^{7}+6.371\times10^{6})\times(3\times10^{8})^2}} \\ &\simeq 0.999999999916665 \end{aligned}$$

So, when a clock on the Earth runs $24\times3600=86400$ s (i.e. one day) an identical clock on an orbiting satellite at 20200 km above the surface of the Earth will run:

$$0.999999999916665\times24\times3600\simeq86399.9999927999\,\text{s}$$

This means that the clock at 20200 km above the surface of the Earth is behind the clock on the Earth by about 7.2 μs per day. In terms of distance (using c as a conversion factor) this time difference is equivalent to:

$$0.000007200113\times3\times10^{8}\simeq2160\,\text{m per day}$$

which is significant.
Comment: apart from the question marks about the validity of the application of special relativity in GPS (see the text), this method of calculation is questionable from another perspective that is the speed of the GPS satellites is calculated relative to the center of the Earth and not relative to the surface of the Earth where the receivers are located.[208] So, if the alleged observed special relativistic time dilation agrees with the result of this method of calculation (assuming it can be distinguished from the general relativistic time dilation at least implicitly within the combined effect) then this should cast more doubts on the validity of the application of special relativity in GPS (and hence on the validity of the application of general relativity in GPS).

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[206] This question and its answer are based on the approach that is followed in the literature of special relativity in dealing with kinematical time dilation. As we noted earlier, we have several reservations on this approach. We should also note that the answer is based in part on classical formulation which can be justified by being a good approximation as well as being related to gravitational (rather than special relativistic) aspect.

[207] Because we are using classical formulation within a special relativistic context, r is metrical.

[208] Due to the rotation of the Earth around its axis, the surface of the Earth is moving relative to its center. No acceptable approximation can explain the neglect of this relative speed. The relative speed should also be affected by other factors.

3. Considering the results of the previous two exercises, find the required combined correction due to relativistic effects.[209]

   Answer: The results show that the clock on the satellite will be in advance by about 45.7 μs per day (due to gravitational effect) and behind by about 7.2 μs per day (due to kinematical effect) and hence a combined relativistic correction of about $45.7 - 7.2 = 38.5$ μs per day is needed. In terms of distance (using c as a conversion factor) this time correction is equivalent to about 11.5 km per day, which is considerable.

   Note: the GPS requires corrections for other effects (gravitational and non-gravitational and relativistic and non-relativistic) and this should introduce more sources of uncertainty on the results used in the GPS test. We should also note that the calculation of the above combined correction is based on the validity of additivity (or superposition) of special and general relativistic effects which is not obvious.

9.4 Gravitational Frequency Shift

1. Justify the fact that the correlation between gravitational time dilation and gravitational frequency shift should diminish the value of the evidence related to these effects in endorsing general relativity.

   Answer: This is because these evidence belong to a single aspect of general relativity (i.e. the effect of gravity on the flow of time) and not to two different aspects and hence instead of having two types of evidence that support two different aspects of the theory we will have only one type of evidence that supports a single aspect of the theory. Moreover, because gravitational frequency shift can be explained classically (e.g. by energy conservation) the correlation means that in many cases even gravitational time dilation may be explained (indirectly) by classical gravity.
2. What is the significance of having a classical formulation for the gravitational frequency shift that is identical to the general relativistic formulation under a power series approximation that is acceptable in most cases?

   Answer: The significance is that the two formulations produce virtually identical predictions in most cases and hence most of the claimed evidence of gravitational frequency shift are not conclusive in their endorsement to general relativity specifically because they can be explained and justified classically. This should also diminish the significance of the gravitational time dilation evidence (at least in some cases) due to the correlation between gravitational time dilation and gravitational frequency shift in general relativity (and hence even time dilation may be explained classically although indirectly).
3. If gravitational time dilation and gravitational frequency shift are equivalent then why should we investigate these effects (in this chapter and in the previous chapters) separately?

   Answer: We note the following:

   - Although these effects are generally equivalent in general relativity they may not be so in other theories (particularly in classical physics where time dilation is meaningless). In fact, even in general relativity the equivalence may be questionable in some cases.
   - Timing mechanisms may not involve frequency and hence the equivalence in such cases is meaningless.
   - From practical and terminological perspectives, time dilation may not be based on actual frequency shift (i.e. by shifting the frequency of a single signal due to its movement from one potential level to another) such as in the case of generating two independent signals at two different potential levels. Also, the experimental techniques differ where the observed effects in some techniques are more appropriate to label as time dilation while in other techniques the effects are more appropriate to label as frequency shift.
4. Discuss the conventional nature of the physical definition and determination of time.

   Answer: Whether time is defined and determined by a repetitive process or by a non-repetitive process the definition is essentially a matter of convention and hence its physical significance is also largely conventional. For example, if we define time flow by the repetitive process of the swing of a pendulum we in fact assume (or rather adopt a convention) that this repetitive process occurs in the same amount of time (i.e. the periods of all cycles are equal). This similarly applies to the definition of time by the atomic transitions and the frequency of radiation. Accordingly, there is no fundamental physical necessity of the consistency of these definitions. For example, there is no fundamental physical necessity that time flow as defined by the pendulum swing is identical (or linearly or proportionally correlated) to time flow as defined by the atomic transition. Also, there is no necessity that these timing definitions and procedures should agree in all cases and circumstances. Yes, we should choose our different timing definitions and procedures such that they produce physically sensible and consistent results and predictions in all cases and circumstances.[210] However, there is always a risk of confusing the physical aspects with the conventional aspects of time and timing procedures. In our view, many issues about time and timing in modern physics are not investigated properly and thoroughly from this perspective and this could be a major reason for many absurdities and paradoxes of modern physics (especially in the relativity theories).

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[209] In fact, the question is about the main (not all) relativistic effects (i.e. the effects investigated in the previous exercises).

Accordingly, we may claim that there is no universal natural process for calibrating and measuring time and hence some of the claimed effects of time dilation (and its alike) may be process dependent (regardless of the reality or non-reality of time). So, to develop a real and invariant physical laws that reflect the physical processes in all cases and circumstances (i.e. independent of the adopted calibration and measurement procedures) we may need to use physical parameters other than time in formulating the physical laws. In fact, this should be inline with the tentative premise that time is not a real physical parameter, i.e. it is a mental process of the observer brain and is influenced by his conventions and definitions and hence it may represent a subjective feeling and experience more than an objective physical reality (or at least it has a subjective as well as objective side). Anyway, this should not affect the reality of absolute frame (especially in its spatial side) which can still exist as a unique physical entity relative to which time and timing (regardless of their nature) can be defined, regulated and calibrated. In fact, the view that time has a subjective as well as objective side (which seems to be the most logical and sensible view) should address this potential difficulty in any case.

9.4.1 Gravitational Red Shift from Astronomical Observations

9.4.2 Pound-Rebka Experiment

1. Derive a simple classical formula for the expected frequency shift in the Pound-Rebka experiment.

   Answer: We should remark first that the following derivation is not entirely classical due to the use of Poincare's mass-energy relation (which may not be classical; see B4) and Planck's energy-frequency relation. However, it is not based on general relativity and hence it is classical in its gravitational aspect (which is the important aspect in this context).

   According to classical physics, the change in the gravitational potential energy of an object of mass m in moving across a height H near the surface of the Earth is mgH where g is the magnitude of the gravitational field (or gravitational acceleration) near the surface of the Earth. Now, a photon of frequency ν has energy $E = h\nu$ according to Planck's formula (where h is Planck's constant) and effective mass $m = E/c^2$ according to Poincare's formula (where c is the characteristic speed of light). Accordingly:

$$\begin{aligned}
\Delta E_f &= \Delta E_p \\
h\Delta\nu &= mgH \\
h\Delta\nu &= \frac{E_f gH}{c^2} \\
h\Delta\nu &= \frac{h\nu gH}{c^2}
\end{aligned}$$

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[210] In fact, the consistency condition is commonly assumed (rather arbitrarily) without investigation or verification and this may be justified by the overall consistency of the physical theories. However, the reality is that even this latter consistency is questionable and can be challenged in many details (as we do frequently). Strangely, the enthusiasm of relativists about the relativity of time is not matched by a similar enthusiasm about the conventional nature of its definition and determination, e.g. they generally state that contraction or dilation of time according to one process or timing mechanism means the same in any other process and mechanism, which gives an absolute and real sense to time.

$$\Delta\nu \quad = \quad \frac{gH}{c^2}\nu$$

where line 1 is based on the conservation of energy (with ΔE_f being the change in the magnitude of the kinetic energy due to change in frequency and ΔE_p is the change in the magnitude of the gravitational potential energy due to change in height) while the other lines are based on the formulae that were given earlier in this answer. We note that in line 3 we are assuming that the effective mass corresponds to the kinetic energy only which may be challenged.[211] However, this is sensible because we are considering the mass that contributes to the potential energy and hence it should not include the potential energy itself. Anyway, this can also be regarded (argumentatively) as a good approximation since the kinetic energy of the photon is generally much larger than its potential energy in the gravitational field of the Earth. For example, a gamma ray photon of frequency 3.4×10^{18} Hz (similar to what is used in the Pound-Rebka experiment) has kinetic energy of about 2.25×10^{-15} J and potential energy at the surface of the Earth of magnitude 1.57×10^{-24} J.

2. Referring to the answer of the previous exercise, justify why high-frequency radiation (like gamma ray) should be desirable to use in experiments like the Pound-Rebka experiment.
   Answer: From the answer of the previous exercise we have $\Delta\nu = \frac{gH}{c^2}\nu$, i.e. the frequency shift $\Delta\nu$ is proportional to the frequency of radiation ν. So, to make the frequency shift sufficiently large to be easily detected and measured (with minimal relative error) we need to use radiation of very high frequency and from this perspective gamma ray is the ideal radiation because it has the highest frequency of the electromagnetic spectrum.
   Note: this exercise and its answer are related to the formal aspects of the derived formula in the previous exercise. However, there are other technical aspects for the choice of iron ^{57}Fe isotope (which is used in the Pound-Rebka experiment as a source for the gamma ray).[212] The interested reader should refer to the literature for details.
3. Use the available data about the Pound-Rebka experiment with the classical formula that was derived in exercise 1 to estimate the frequency shift and the fractional frequency shift in the Pound-Rebka experiment.
   Answer: In the Pound-Rebka experiment the height of the tower is $H \simeq 22.5$ m and they used iron ^{57}Fe isotope which emits gamma rays of frequency $\nu \simeq 3.4 \times 10^{18}$ Hz. Hence, the frequency shift according to the classical formula is:

$$\Delta\nu = \frac{gH\nu}{c^2} \simeq \frac{9.8 \times 22.5 \times 3.4 \times 10^{18}}{(3 \times 10^8)^2} \simeq 8.3 \times 10^3\,\text{Hz}$$

   while the fractional frequency shift is:

$$\frac{\Delta\nu}{\nu} = \frac{gH}{c^2} \simeq \frac{9.8 \times 22.5}{(3 \times 10^8)^2} \simeq 2.5 \times 10^{-15}$$

   Note: in these calculations we are assuming that the rays are descending in the gravitational well and hence the frequency shift is positive (i.e. blue shift). In fact, this is consistent with the setting of one version of this experiment where the source of gamma rays was placed at the top of the tower while the detector was placed at the bottom of the tower. Alternatively, we can assume that we are using the magnitude of the frequency shift and hence the calculations can represent both versions of this experiment.
   We should also remark that the values used in the calculations of this exercise (as well as in the next two exercises) are mainly for the purpose of demonstration and comparison and hence they may not be identical to some of the reported values in the literature (especially those related to the early versions of the Pound-Rebka experiment which have more sources of uncertainty).
4. Repeat exercise 3 using this time the general relativistic formula for the gravitational frequency shift that was derived in § 6.3.4. Comment on the results.

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[211] The negative nature of the potential energy should not affect the legitimacy of posing this assumption and its challenge.

[212] This is due to the Mossbauer effect which is exploited in the detection technique of this experiment.

Answer: We subscript the variables on the ground with 1 and the variables at 22.5 m above the ground with 2. Let assume that the frequency at the ground is $\nu_1 \simeq 3.4 \times 10^{18}$ Hz. According to the general relativistic formula that was derived in § 6.3.4 we have:

$$\begin{aligned}
\nu_2 &= \nu_1 \left(1 - \frac{2GM}{c^2 r_1}\right)^{1/2} \left(1 - \frac{2GM}{c^2 r_2}\right)^{-1/2} \\
\nu_1 - \nu_2 &= \nu_1 - \nu_1 \left(1 - \frac{2GM}{c^2 r_1}\right)^{1/2} \left(1 - \frac{2GM}{c^2 r_2}\right)^{-1/2} \\
\frac{\nu_1 - \nu_2}{\nu_1} &= 1 - \left(1 - \frac{2GM}{c^2 r_1}\right)^{1/2} \left(1 - \frac{2GM}{c^2 r_2}\right)^{-1/2} \\
\frac{\Delta\nu}{\nu_1} &= 1 - \left(1 - \frac{2GM}{c^2 r_1}\right)^{1/2} \left(1 - \frac{2GM}{c^2 r_2}\right)^{-1/2} \\
\frac{\Delta\nu}{\nu_1} &\simeq 1 - \left(1 - \frac{2 \times 6.674 \times 10^{-11} \times 5.972 \times 10^{24}}{(3.0 \times 10^8)^2 \times 6.371 \times 10^6}\right)^{1/2} \times \\
&\quad \left(1 - \frac{2 \times 6.674 \times 10^{-11} \times 5.972 \times 10^{24}}{(3.0 \times 10^8)^2 \times (6.371 \times 10^6 + 22.5)}\right)^{-1/2} \\
\frac{\Delta\nu}{\nu_1} &\simeq 2.4 \times 10^{-15} \\
\Delta\nu &\simeq 8.3 \times 10^3 \,\text{Hz}
\end{aligned}$$

Comment: on comparing the results of exercise 3 with the results of this exercise we can see that the classical and general relativistic formulations give virtually identical results and hence the results of the Pound-Rebka experiment are not conclusive in endorsing general relativity specifically.

5. Repeat exercise 3 using this time the classical formula for the gravitational frequency shift that was derived in § 8.4. Comment on the results.

Answer: We subscript the variables on the ground with 1 and the variables at 22.5 m above the ground with 2. Let assume that the frequency at the ground is $\nu_1 \simeq 3.4 \times 10^{18}$ Hz. According to the classical formula that was derived in § 8.4 we have:

$$\begin{aligned}
\nu_2 &= \nu_1 \left(1 - \frac{GM}{c^2 r_1}\right) \left(1 - \frac{GM}{c^2 r_2}\right)^{-1} \\
\nu_1 - \nu_2 &= \nu_1 - \nu_1 \left(1 - \frac{GM}{c^2 r_1}\right) \left(1 - \frac{GM}{c^2 r_2}\right)^{-1} \\
\frac{\nu_1 - \nu_2}{\nu_1} &= 1 - \left(1 - \frac{GM}{c^2 r_1}\right) \left(1 - \frac{GM}{c^2 r_2}\right)^{-1} \\
\frac{\Delta\nu}{\nu_1} &= 1 - \left(1 - \frac{GM}{c^2 r_1}\right) \left(1 - \frac{GM}{c^2 r_2}\right)^{-1} \\
\frac{\Delta\nu}{\nu_1} &\simeq 1 - \left(1 - \frac{6.674 \times 10^{-11} \times 5.972 \times 10^{24}}{(3.0 \times 10^8)^2 \times 6.371 \times 10^6}\right) \times \\
&\quad \left(1 - \frac{6.674 \times 10^{-11} \times 5.972 \times 10^{24}}{(3.0 \times 10^8)^2 \times (6.371 \times 10^6 + 22.5)}\right)^{-1} \\
\frac{\Delta\nu}{\nu_1} &\simeq 2.4 \times 10^{-15} \\
\Delta\nu &\simeq 8.3 \times 10^3 \,\text{Hz}
\end{aligned}$$

Comment: the three formulations (i.e. two classical and one general relativistic) in the last three exercises produce virtually identical quantitative results and this should diminish the value of the Pound-Rebka experiment in endorsing general relativity specifically and definitely. Moreover, the quantitative agreement between the two classical formulations (of exercise 3 and this exercise) should reveal that these formulations are essentially equivalent.

9.4.3 Gravity Probe A

9.5 Gravitational Length Contraction

1. Try to justify the claim that observing gravitational length contraction is not as easy as observing gravitational time dilation (and it may even be impossible).
   Answer: One potential reason is that we can store the cumulative effect of time dilation but not length contraction. To be more clear, in time dilation we can take two synchronized clocks and put them in two positions in space and after a while we bring them together in one position and compare the time intervals they recorded.[213] But this is not possible in length contraction because if we put two identical sticks in two positions in space and one is contracted relative to the other then as soon as we bring them together for comparison they should restore their state as being identical because the effect of length contraction is position-dependent and hence it will be revoked as soon as the sticks are reunited. So, the key difference between the two effects is that time intervals can be stored in clocks where they can be compared by bringing the clocks together while lengths cannot be stored in sticks to be compared by bringing the sticks together.
   However, the above justification for the difficulty or impossibility of observing gravitational length contraction is rather naive because it is based on a very crude and direct method of observation of this effect and hence we cannot rule out the possibility of observing this effect indirectly and through its implications and consequences using potentially sophisticated methods.
   We should remark that there are more elaborate and sophisticated arguments in support of the impossibility of observing gravitational length contraction than the above argument. However, these arguments can also be challenged and refuted. We should also note that there are claims that gravitational length contraction belongs to the space and not to the objects in space (similar to the alleged cosmological expansion) and hence it is not observable.[214] However, this can be easily challenged by the correspondence with the kinematical length contraction of special relativity where length contraction supposedly belongs to objects (or at least it is observable in objects) and hence it should be so in general relativity. In fact, we can ask: if gravitational length contraction does not affect objects then what are its physical effects and consequences and why is it different from time dilation where their status in the metric (as represented for example by the Schwarzschild metric) is very similar?
2. What is the important issue that is highlighted by the previous exercise?
   Answer: The previous exercise highlights the issue that the spacetime of general relativity is not as symmetric in its spatial and temporal dimensions as it should be. In other words, the rules that apply to the temporal dimension are not necessarily the same as the rules that apply to the spatial dimensions. One demonstration of this issue is the fact that all the alleged evidence and verifications that are related to the nature of spacetime of general relativity are essentially about time and this should indicate that time and space are still distinct and they differ in their properties despite the presumed merge in spacetime.

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[213] The description of this procedure is rather loose. However, it can be made rigorous.

[214] We note that the analogy with the cosmological expansion is inappropriate (apart from the superficial similarity) because in the cosmological expansion the expansion of objects is supposed to be hindered by gravitation and internal forces while in the presumed gravitational length contraction the contraction (or dilation) is supposed to be a direct consequence and effect of the gravity itself so gravity is a cause and driver rather than a hindrance to this effect.

9.6 Gravitational Waves

9.6.1 Indirect Observation of Gravitational Waves

1. Why should the emission of gravitational waves by a binary orbiting system shorten the orbital period?
Answer: For simplicity, let use classical physics whose fundamental principles are valid in this explanation (although historically gravitational waves are not a classical prediction). From the well known classical relation (which may be labeled as virial theorem) in orbiting systems we have $2K + U = 0$ where K and U are the kinetic and potential energy of the orbiting system. We also have $E = K + U$ where E is the total energy of the system. Hence, we get:
$$E = -\frac{1}{2}U + U = \frac{1}{2}U = -\frac{Gm_1m_2}{2r}$$
where m_1 and m_2 are the masses of the (binary) orbiting system. The last relation means that r decreases/increases as E decreases/increases (noting the minus sign).[215] Now, according to the Kepler's third law we have $T^2 = \frac{4\pi^2r^3}{GM}$ where T is the orbital period of the system and r is the mean distance.[216] This relation means that T decreases/increases as r decreases/increases. By combining these two facts, we conclude that T should decrease/increase as E decreases/increases. Now, in a binary orbiting system that continuously emits gravitational waves E should decrease (since emission of gravitational waves means losing energy) and hence we conclude that the orbital period of this system should continuously decrease.
2. Assess the claimed evidence of indirect observation of gravitational waves through the detection of changes in the orbital period of binary system.
Answer: We may note the following:
• Apparently, there is only one instance of this indirect test (namely the PSR B1913+16 binary pulsar system) and hence the indirect test is not sufficiently endorsed. For example, a single error somewhere in the observation or analysis of this system could damage the entire test and make it useless.
• The significance of this test relies on the ability of the linearized general relativity to endorse the non-linearized general relativity and this can be strongly challenged.
• The observations and their analysis are shrouded with many theoretical and practical uncertainties and potential errors.
• The theory and calculations which this test is based on are partially classical (at least by using flat-space physical variables) and hence its endorsement to general relativity (even as an approximation) should be questioned. The reason is that this is a strong-gravity system and hence a strictly general relativistic treatment is required; moreover any relativistic effects (even if they are tiny) should amplify and become significant by accumulation due to the violation of locality (temporally and spatially).
• Indirect *observations* in general have no quantitative significance from the gravitational waves perspective since no definite quantitative attributes of the alleged waves can be deduced from the results and compared to the theory. Hence, the value of indirect *observations* is at most qualitative from this perspective.
• Being indirect *observation* evidence should diminish its value and significance because there is a possibility that the observed change of periodic time in these indirect *observations* may be caused by mechanism(s) other than emission of gravitational waves, e.g. by the emission of electromagnetic waves assuming the binary system is charged or by the existence of a dissipating frictional mechanism. So, as long as there is no actual detection of gravitational waves and genuine observation of their properties we cannot be certain about the emission of gravitational waves (and hence the test could be entirely baseless, qualitatively as well as quantitatively).
• Gravitational waves can be predicted (at least qualitatively) by theories other than general relativity and hence the test is not conclusive in endorsing general relativity specifically. In fact, some metric

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[215] In fact, this technical argument is not really needed because we can simply argue that the binary system is a bound system and hence its total energy is negative where this total energy tends to zero (i.e. increases) as the two objects separate more and more.

[216] Even if we assume circular orbit the generality of the result should not be affected.

gravity theories should provide similar quantitative predictions to the prediction of general relativity and hence this alleged evidence is not specific to general relativity even quantitatively (assuming it has quantitative significance).
Accordingly, this test in our view is not as significant and conclusive as it is claimed by general relativists.

9.6.2 Direct Observation of Gravitational Waves

1. Provide a brief description of LIGO. Also, list some of the alleged observations of gravitational waves made by LIGO.
   Answer: LIGO is a US-based scientific facility that is designed to detect gravitational waves generated by cataclysmic astronomical events such as explosion of supernovae and collision of neutron stars and black holes. LIGO employs two huge and widely-separated gravitational wave detectors: one based in Hanford (Washington) and one based in Livingston (Louisiana). These detectors, which work in unison, use laser interferometric techniques to detect and measure the tiny fluctuations in spacetime caused by the passage of gravitational waves. LIGO is the largest Earth-based gravitational wave observatory.
   The first alleged detection of gravitational waves by LIGO occurred in September 2015 where this observation was related to the event of collision and merger of two black holes which are about 1.3 billion light years away from the Earth. According to the reports: "the gravitational waves were produced during the final fraction of a second of the merger of two black holes to produce a single, more massive spinning black hole". The first alleged detection was followed by a number of other alleged detections (e.g. on December 2015 and January 2017) of gravitational waves from similar collision and merger events.
2. Provide a brief description of LISA.
   Answer: According to the proposed plan, LISA is a space-based dedicated observatory that uses laser interferometric techniques to make direct detection and measurement of gravitational waves generated by cataclysmic astronomical events. It is made of three spacecraft trailing the Earth in its orbit around the Sun in a state of gravitational free fall. These spacecraft form an equilateral triangle with sides 2.5×10^9 m long.[217] The principle of operation of this triangular formation is based on measuring the differential change in the length of its sides caused by the passage of gravitational waves where this change is detected by Michelson-like laser interferometry. LISA project, which is run by the European Space Agency (ESA), is still under development and test and hence no announcement of detection of gravitational waves is made yet. In fact, even the deployment of the spacecraft according to the above-described formation is not confirmed.[218] We note that the LISA project was originally a collaborative effort between ESA and NASA but NASA withdrew in 2011 from the project due apparently to funding difficulties (and possibly to the realization of non-feasibility and immense technical difficulties).
3. Assess the recent claims of detecting gravitational waves by LIGO.
   Answer: In our view, it is unwise to believe that the announced observations of gravitational waves by LIGO are credible and conclusive. A main source of suspicion in the alleged observations is that even if we accept that the detector is sufficiently sensitive to detect gravitational waves, it is difficult to imagine that all sources of noise are neutralized or eliminated correctly in the analysis. In fact, there are many sources of noise as well as sources of uncertainty related for example to the source of gravitational waves and its spatial determination. We should remark that the significance of the LIGO observations (or at least some of them) was questioned by a number of respected scientists on the basis of the validity of the method used to distinguish the gravitational wave signal from the background random noise. The interested reader is referred to the literature for details.
4. Assess the value of the experimental projects (such as LIGO and LISA) about the detection and observation of gravitational waves.
   Answer: In our view, these projects are typical examples of the lack of judgment and proper sense

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[217] There is some conflict in the literature about the proposed dimensions of LISA (so the quoted figure is not definite).

[218] It is reported that the launch of LISA is scheduled for 2020.

of direction in modern physics and the reckless use of resources in effectively futile projects. A few of our arguments against these projects (and their alike) are given in the following points:

- These lavish projects consume a huge amount of resources of all kinds with very little (if any) actual return. In fact, even if these projects have real outcomes, they should be put at the very bottom in the priority list since there are many much more essential scientific (as well as non-scientific) projects that desperately need these resources.[219]
- There are more important (and seemingly easier to detect) consequences of general relativity to look for; in fact these alternatives should be even much cheaper to investigate and have more significance in endorsing general relativity or otherwise. For example, according to the fanciful journey of a falling massive object toward a black hole (see § 8.7) a collapsing object (such as a star) to become a black hole should have a distinctive radiation signature which should be much easier to detect than the aforementioned gravitational waves "produced during the final fraction of a second of the merger of two black holes". Another similar example is the search for direct or indirect consequences of the alleged gravitational length contraction. In fact, there are many untouched (practical and theoretical) aspects in general relativity that are much more worthwhile to investigate than these alleged gravitational waves and their alike.

5. Discuss briefly the "elastic body" method for the direct detection of gravitational waves.
   Answer: This method is commonly described as resonant detection method because the detection of waves in this method relies on the magnification of the amplitude of the induced vibrations in the metal bars (which are commonly known as Weber bars) by resonance. Accordingly, the design of the bars takes this into account where the natural resonance frequency of the bars is made (by design) to match the frequency of the anticipated gravitational waves. In fact, this should be a limitation in this method (as compared to the free particle method) since the design of the bar detector is wave-specific which grants less freedom in choosing the type of waves that to be targeted for observation. From a historical perspective the resonant detection method is the older of the two methods. However, it is inferior relative to the free particle method due to its poor sensitivity and wave-specificity although it is generally simpler, cheaper and less resource-demanding than the free particle method (and for these reasons it is still in use these days in its modified and improved versions).

9.7 Black Holes

1. Assess black holes as evidence for general relativity.
   Answer: There are two aspects to this test: a qualitative aspect (i.e. the existence of black holes) and a quantitative aspect (i.e. the details of black holes and their physics). The former aspect is not conclusive in its endorsement to general relativity since the existence of black holes can be classically justified, while the latter aspect has not been endorsed by experimental or observational evidence. In brief, black holes can be easily accommodated within the framework of classical physics (as well as other gravity theories) and hence their existence (if proved) is not a conclusive evidence for general relativity. Yes, there are some mathematical artifacts of the theory of black holes that are specific to general relativity (because they are derived specifically from the formalism of this theory), but none of these artifacts is proved or observed (although there are allegations of the existence of circumstantial evidence in support of some of these artifacts). So, within the limits of our knowledge about black holes, black holes do not represent any conclusive evidence or verification to the theory of general relativity.
2. Assuming Hawking radiation, calculate the power radiated by a black hole of one Earth mass and comment on the result.

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[219] This reminds me of the generous "charitable" donations to rebuild Notre Dame Cathedral while millions of people around the Globe suffer from malnutrition (and even starvation) and have no access to clean water. In fact, this reflects the miserable state of humanity and the widespread hypocrisy and lack of sincerity and wisdom in all aspects of life.

Answer: The power radiated by such a black hole is given by Eq. 47, that is:

$$\begin{aligned}\frac{\sigma\hbar^4c^8}{256\pi^3k^4G^2M^2} &\simeq \frac{5.67037\times10^{-8}\times\left(1.05457\times10^{-34}\right)^4\times\left(3\times10^8\right)^8}{256\pi^3\times(1.38065\times10^{-23})^4\times(6.674\times10^{-11})^2\times(5.972\times10^{24})^2}\\ &\simeq 1.00426\times10^{-17}\ \text{J/s}\end{aligned}$$

Comment: Hawking radiation of such a black hole (which is supposed to be miniature and hence its radiation should be easier to observe) cannot be observed (at least by the means and capabilities of current science) and hence its existence and non-existence (and possibly the existence and non-existence of the black hole itself) is irrelevant to science. We note that since the power radiated by a black hole is inversely proportional to the mass squared the detection of Hawking radiation from stellar and super-massive black holes is much harder than the detection of Hawking radiation from miniature black holes and hence it should be impossible. Yes, the detection of Hawking radiation from the very small miniature black holes should be possible but the existence of such black holes is no more than science fiction.

9.8 Geodetic Effect

9.8.1 Precession of "Moon-Earth Gyroscope" in Motion around Sun

9.8.2 Gravity Probe B

9.9 Frame Dragging

9.9.1 LAGEOS Satellites

9.9.2 Gravity Probe B

9.10 Wormholes and Other Fantasies

9.11 Cosmological Predictions

9.12 Tests of the Equivalence Principle

1. Assess the value of the equivalence principle tests as endorsing evidence to general relativity.
   Answer: A major challenge to these tests is that most (if not all) these tests are related to the classical equivalence principle and hence they cannot be regarded as evidence in support of the weak equivalence principle (due to the difference between the classical and weak equivalence principle; see § 1.8.2) let alone the strong equivalence principle. Moreover, the validity of the principle does not imply the validity of the theory. Anyway, there is no credible claim of the existence of evidence in support of the strong equivalence principle.
   Note: there are claims of evidence (generally indirect) or rather indications in support of the strong equivalence principle from the experimental tests of the equivalence principle. However, apart from being questionable in their validity and significance, they cannot validate the generality of the strong equivalence principle due to verified violations of its validity in certain physical situations (see for example § 10.1.1). In fact, without general validity the strong equivalence principle will have no predictive value and hence it becomes effectively invalid (or at least useless).

9.13 Tests of Special Relativity

9.14 Circumstantial Evidence

9.15 Evidence for Newtonian Gravity

9.16 Final Assessment

1. Discuss the main types of evidence of general relativity and assess their value and significance.
   Answer: We may classify the main types of evidence as follows:
   - Some of the claimed evidence are based on tuning the theory to match the observations (e.g. perihelion precession of Mercury). This obviously is not a prediction of the theory and hence its value is diminished by this factor.
   - Some of the claimed evidence are based on predictions that are not specific to general relativity since they can be explained by other theories based for example on classical physics (e.g. the existence of black holes or gravitational frequency shift). Hence, they cannot be used to endorse general relativity specifically.
   - Some of the claimed evidence are based on trust. Although science in general is based on trust we have strong reasons to have less trust in the claimed evidence in support of general relativity due to the huge bias in favor of Einstein and his theories and the strong desire of many scientists to support these theories. In fact, the 1919 solar eclipse expedition of Eddington and his team demonstrates that trust in science and scientists is not always a good practice. We believe that there are many instances of Eddington and his scientific approach and ethical standards in science especially in the field of relativity theories.
   - Some of the claimed evidence do not require the full formulation of general relativity (e.g. gravitational time dilation which can be obtained, at least qualitatively, from the equivalence principle) and hence they do not represent conclusive evidence in support of the theory as a whole although they can be evidence for some of the principles which the theory rests upon.
   - Some of the claimed evidence are not independent since they rely in their working principle on other types of evidence (e.g. gravitational frequency shift which can be correlated to gravitational time dilation) and hence they have no value as independent type of evidence.
   - Some of the claimed evidence are based on the use of certain metrics (e.g. Schwarzschild metric) or restricted formulation (e.g. linearized form) and this should limit the validity of the evidence in its significance for endorsing the theory in general because these represent special cases or particular instances of the theory and hence they have no general validity (i.e. with regard to other cases and instances).
   - Some of the claimed evidence are simply useless in endorsing general relativity such as the tests of the equivalence principle or the tests of special relativity or the tests of Newtonian gravity because their validity as evidence for general relativity is baseless.
   - Some of the claimed evidence have considerable uncertainties and large error margins, such as the majority of the light deflection and frame dragging tests, and hence they are not conclusive in endorsing general relativity.
   - Some of the claimed evidence have ambiguities and controversies in their theoretical foundations and formulation, such as Shapiro time delay test, and hence they are not definite in their implication and significance.
   - Some of the claimed evidence are qualitative in nature and hence they do not have rigorous quantitative value. In fact, many (if not most) of the claimed evidence are of this type.

   We should finally note that most of the claimed evidence fall into more than one of the above types, e.g. gravitational frequency shift which can be explained in terms of gravitational time dilation and it can be explained by other theories, or black holes or gravitational lensing which are qualitative in their significance and they can be explained by other theories. The value of such evidence is obviously more degraded due to the involvement of more than one degrading factor.
2. List some general factors that could diminish the value of the evidence in support of general relativity.

Answer: For example:

- The strong tendency in modern physics to support general relativity and the desire of the mainstream physicists to associate themselves with this theory and endorse it should be a decisive factor in the overall assessment of this theory and its evidence. This means that the likelihood of false endorsement, due for example to intentional or unintentional bias, is much higher than the normal (as for example in verifying other theories) and this should diminish the value of many claimed evidence and tests in support of general relativity. So, even some of the "conclusive" evidence could become questionable and hence they require further verifications and checks. In this regard, we may quote the following excerpt from McCausland:[220]

It is also reasonable to ask whether the rapid and strong entrenchment of the general theory that occurred as a result of the eclipse announcement may have led experimenters to try to obtain the "right" answers from their observations, as suggested in the above quotation from Sciama (1969). For example, Collins and Pinch (1993) have commented on the measurements of the red-shift predicted by general relativity as follows: The experimental observations, conducted both before and after 1919, were even more inconclusive. Yet after the interpretation of the eclipse observations had come firmly down on the side of Einstein, scientists suddenly began to see confirmation of the red shift prediction where before they had seen only confusion. (End of quote)

- Even if we assume that all the claimed evidence in support of general relativity are correct and conclusive they just prove certain (and in many cases rather minor) aspects of general relativity (such as the weak equivalence principle assuming that these tests are actually tests to this principle rather than tests to its classical version) but they are not sufficient to prove some of the fundamental aspects of the theory such as the strong equivalence principle. In brief, even if all the claimed evidence are conclusive they are not sufficient to prove the theory as a whole and in all its fundamental aspects and implications because all the claimed evidence are partial and they are related to certain aspects of the theory. There are many aspects of the theory of general relativity that are not endorsed yet by the available evidence. Hence, it is premature to say that the theory has passed all the required tests even if all the claimed evidence are conclusive because the claimed evidence are restricted to certain aspects of the theory.
- Most (or all) of the claimed evidence fall within a certain gravitational regime[221] and hence they are not significant or conclusive with regard to more extreme gravitational regimes. In this regard, we may quote the following excerpt from Chandrasekhar:[222]

It does not seem to me that the successes of Einstein's theory are either long or impressive. It is true that his prediction of the different rates of clocks in locations of differing gravity, his prediction of the deflection of light when traversing a gravitational field and resulting time delay, his prediction regarding the precession of the perihelion of Mercury, and finally, the slowing down of a binary star in an eccentric orbit by virtue of the emission of gravitational radiation, have been confirmed quantitatively. But all these relate to the departures from Newtonian theory by a few parts in a million; and of no more than three or four parameters in a post-Newtonian expansion of Einstein's field equations. And so far, no predictions of general relativity in the limit of strong gravitational fields have received any confirmation; nor are they likely in the foreseeable future. (End of quote)[223]

- Almost all the claimed evidence are related to the gravitational aspects of general relativity and hence even if they are valid and conclusive they can only establish the theory as a gravity theory and not as a "General Theory".
- Almost all the claimed evidence are based on the Schwarzschild metric and hence the endorsing value of the entire body of evidence could be damaged if this metric (or its physical equivalent) proved to be special or can be obtained from other theories or can be derived from some arguments instead of the Field Equation (or can be affected by any other compromising factor).

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[220] See "Anomalies in the History of Relativity".

[221] We may describe this regime as "Newtonian and quasi-Newtonian" where "quasi-Newtonian" represents rather small perturbations to the Newtonian regime.

[222] See "*A Quest for Perspectives* Selected Works of S. Chandrasekhar *With Commentary*".

[223] The last statement may be disputed by claimed subsequent tests.

3. Our assertion that the predictions that are not specific to general relativity (by being classically explicable for instance) have limited endorsing value may be challenged. Discuss this issue.
   Answer: Our assertion may be challenged by a counter claim that the evidence that can be classically justified should lend full endorsement to general relativity either because it supports the convergence of general relativity to classical gravity in the classical limit which is a requirement of the correspondence principle or because its value in endorsing general relativity cannot be diminished by being classically justifiable (or justifiable by another theory). However, we may challenge this by the following:
   - To be a requirement of the correspondence principle, we need to show formally (rather than by just qualitative or quantitative agreement) that general relativity agrees with the classical predictions in these particular classical systems. So, when the evidence can be justified by another theory (whose compliance with general relativity is not established formally) its endorsement to general relativity through convergence to the other theory in these particular classical systems is questionable although its support to general relativity may still be claimed independently.
   - The claim that the evidence that is common to both general relativity and another theory has full endorsing value to general relativity should be less logical (if not wrong) when the two theories do not have identical or equivalent predictions because the evidence cannot support two conflicting theories (even if this conflict is related to another aspect of the theories).
   - We may even claim that the accumulation of evidence of this type (i.e. evidence that can be explained classically) should not only diminish the value of these evidence but even the value of general relativity in its entirety and generality because this shows that general relativity is mostly supported in classical systems and hence its most important (i.e. non-classical) implications are either not supported or weakly supported. Furthermore, general relativity will be like a redundant theory since similar explanations and justifications can be obtained from classical gravity (or another theory) and hence the value of such predictions (as *general relativistic* predictions) should be diminished. We may contrast this with quantum mechanics for example where no credible theory can replace it in explaining and justifying the quantum phenomena.
4. Why factors like fraud and error should be considered more seriously in assessing the evidence in support of general relativity?
   Answer: Fraud and error are common in science in general. However, they are more likely to occur in general relativity than in other theories such as quantum mechanics because there are many motivating factors to endorse general relativity (some of these factors are discussed earlier). Moreover, many of the alleged evidence in support of general relativity are obtained in very exceptional and exotic physical systems and circumstances. In fact, some of these systems and circumstances are on the very edge of the capability of modern science and technology and hence they are very special and have no presence in practical life. Accordingly, investigating their physics is very delicate (such as verifying an effect that is extremely small like gravitational waves or frame dragging) and hence it is susceptible to many sources of uncertainty and error. This is unlike quantum mechanics for example where it has direct and common applications that we all feel everywhere in our daily life.[224] In brief, the likelihood that the evidence of general relativity is fabricated or distorted or exaggerated or contaminated with large errors and uncertainties is higher than usual and hence its acceptance requires more trust and reliance on the integrity of scientists and their methods, analysis, equipment and judgment. Therefore, on a purely statistical basis the claimed evidence in support of general relativity is more likely to be fabricated or exaggerated or erroneous or inconclusive and hence it should be scrutinized in an exceptional way.

9.17 Evidence against General Relativity

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[224] Due to several reasons that we discussed in § 9.3.4, global positioning system cannot be compared to the applications of quantum mechanics in our daily life.

Chapter 10
Challenges and Assessment

10.1 Challenges and Criticisms

10.1.1 Limitations and Failures of the Equivalence Principle

1. Discuss the failure of the alleged equivalence between inertial and gravitational mass in achieving this equivalence.[225]

   Answer: One of the main objectives at the start of the development of general relativity was to explain the equivalence between inertial mass and gravitational mass (which allegedly has no explanation in the framework of classical physics) and accordingly the equivalence principle was developed and considered as one of the main pillars of general relativity. However, in the end of this development the theory seems to have lost the aforementioned objective and hence the equivalence between inertial mass and gravitational mass remained in need for an explanation (possibly more desperately than at any time before). The reason is that according to general relativity mass is the source of gravity (or rather the cause of the spacetime geometry) and hence we initially seem to have "gravitating" mass. However, since gravity is considered an effect of this geometry that is demonstrated in following geodesic trajectories (which are independent of mass) in the spacetime, there is no need for "gravitated" mass. This becomes more obvious if we note that gravitated massless objects follow geodesic trajectories (where this behavior is supposed to be the essence of gravitation assuming the absence of other forces) as much as gravitated massive objects follow geodesic trajectories and hence no mass is required for gravity to take place.[226] In brief, in general relativity we seem to have "gravitating" mass but we do not have gravitated mass.

   In fact, we can take this argument further by claiming that even "gravitating" mass (as such) does not exist in general relativity because we do not have gravitation at all when we consider this "gravitation" as mere geometric attribute determined by geodesic tracks in spacetime. This should be consolidated by the fact that gravitation is a binary relation between two objects (i.e. gravitating object and gravitated object) and hence if we do not have gravitated mass we cannot have gravitating mass. Accordingly, the minimum that we can claim is that there is no gravitated mass and the maximum is that there is no gravitational mass (whether gravitating or gravitated). So, if we do not have gravitational mass then what is the meaning of the equivalence between inertial mass and gravitational mass. In fact, even if we lost only the gravitated mass we still cannot have a sensible meaning for the equivalence between inertial mass and gravitational mass since this equivalence depends on the existence of gravitational mass in the form of gravitated mass and this should be obvious from inspecting the equivalence principle and analyzing its logic (see § 1.8.2). Actually, this is an epistemological disaster because we are supposed to justify and explain the equivalence by what leads to denying this equivalence (since we abolished one participant in this equivalence).

   The above reasoning may seem an elaborate verbal argument with no substantial physical content. But this is not true because the physical (as well as logical) absurdity of negating the gravitational mass while trying to justify its equivalence to the inertial mass is obvious. Moreover, this argument is deeply linked to the issue of absolute frame and the justification of the existence of inertial frames and their distinction from non-inertial frames. In other words, when we have no absolute frame and we have no gravitating mass (which may refer to the background cosmic distribution according to a potential interpretation of the Mach principle) then we should have no inertial frames (because they are meaningless or at least because they are inexplicable) and hence we should have no special relativity which supposedly underlies general relativity. Although this seems to be easily revoked by claiming that "the spacetime of special relativity is an idealization that does not exist", it is not. First, because this alleged idealization is not really an idealization; otherwise the validity of the whole special relativity (as a realistic theory that belongs to the real world and supported by physical evidence) will be questionable.[227] Second, this "idealization" is at least a valid approximation which is at the foundation of general relativity and its reliance on the local application of special relativity.

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[225] As discussed earlier (see § 1.8.2), the alleged duality of inertial mass and gravitational mass is a general relativistic issue and not a classical issue since in classical physics mass is mass whether it is in Newton's second law or in Newton's gravity law. Accordingly, this exercise is about a challenge to general relativity specifically.

[226] The requirement of gravity to mass (and if *massless* objects are really massless) was investigated thoroughly earlier.

2. According to the equivalence principle of general relativity, a stationary observer relative to a source of gravity (e.g. a terrestrial lab) is equivalent to an accelerating observer. Analyze and discuss this proposition (which should lead to another failure of the equivalence principle).

   Answer: If we accept the (generally-accepted) proposition in general relativity that an accelerating observer can be represented by a series of instantaneous inertial rest observers, then the proper time of an accelerating observer should dilate (i.e. kinematically) continuously (i.e. it slows down in an increasing rate) relative to any inertial observer. However, this is not the case for the stationary observer relative to a source of gravity where the time of this observer should dilate (i.e. gravitationally) in a fixed rate relative to an inertial stationary observer at infinity for example. This should put a question mark on the validity of the equivalence principle and the validity of the local application of special relativity in the spacetime of general relativity.

   Note 1: in our view (which may be disputed), the premise that an accelerating observer can be represented by a series of instantaneous inertial rest observers (when this premise is applied in general relativity) should be ultimately based on the strong equivalence principle where special relativity is supposed to apply in the local inertial frames that coordinate the spacetime of general relativity.

   Note 2: the above is just an example of many other examples of the failure of the equivalence principle (e.g. the electric charge example given in the text). These failures are generally attributed to the locality restriction as discussed in the text. However, we believe that at least some of these failures have more fundamental reasons than the violation of locality. In fact, many of these failures originate from a fundamental failure of the equivalence itself (i.e. even at local level) due to the difference in nature between gravity and acceleration. Hence, even locality cannot guarantee the equivalence. Also, see next exercise.

3. As part of the extension of special relativity to include the domain of general relativity, an accelerating frame should be represented by a series of instantaneous inertial rest frames (IIRF) where special relativity applies in these frames. Discuss and assess this.

   Answer: This representation (and hence the consequent application of special relativity) cannot be accepted in general. For example, the accelerometer of an accelerating frame should read differently from the accelerometer of the IIRF (i.e. the latter should read zero acceleration while the former should read non-zero acceleration) and this difference in reading is a real physical effect that should have real physical consequences.[228] In fact, this should indicate that the local application of special relativity should not be valid in general and this should put a question mark on the general validity of the equivalence principle of general relativity. The reader should note that the locality restriction of the equivalence principle cannot come to the rescue of this principle and prevent this failure because this difference is felt locally regardless of any extended spacetime (unlike the above example of electric charge for instance where the imposition of the locality condition in a confined region of spacetime can allegedly rescue this principle since the radiation is not a local effect according to the claim).

4. Analyze and assess the local application of special relativity in the spacetime of general relativity according to the strong equivalence principle.

   Answer: As we know, special relativity requires two main conditions for its applicability: the flatness of the spacetime (since special relativity does not include gravity, which supposedly distorts spacetime, or deal with gravitational frames) and the inertiality of the reference frame (since special relativity does not deal with accelerating frames). Now, let see how these conditions are embedded in the strong equivalence principle for the local application of special relativity. The most logical explanation is that the locality condition ensures flatness (because Riemannian space is locally flat) while the free fall of frame ensures inertiality (due to the cancellation of the effect of "uniform" gravity, which is equivalent to acceleration, by the acceleration of the free fall according to the weak equivalence principle). However, this does not seem entirely logical and consistent with the framework of general relativity because according to the principle of metric gravity (see § 1.8.4) gravity is represented by the curvature of spacetime. Now, since the spacetime is locally flat then where is the effect of gravity in this local patch of spacetime, and how the transformation from the freely falling frame to the corresponding local frames represents and incorporates the effect of gravity?[229] This criticism may be tackled by claiming that gravity is represented mainly by the curvature of spacetime on a global level rather than local level, moreover the local flatness of spacetime is not exact (i.e. at local level it is approximately flat). Accordingly, locality ensures "approximate flatness" and hence the function of the transformation from freely falling frame to the corresponding frames is to account for the effect of the residual gravity in the local patch of spacetime. However, this tackling does not seem entirely convincing (for example, what is the role of the "uniform" gravity?).

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[227] In essence, we are claiming that there are certain patches in the physical spacetime (or Universe) where special relativity is valid. This does not mean non-existence of mass but it means that any existing mass has no direct (or local) gravitational role or effect although it may have other effects such as inertial effects (for example by providing an ambient background uniform "gravitational" field inline with a possible interpretation of the Mach principle; see § 6.5.1).

[228] We should note that this difference applies to physical factors other than the reading of accelerometers. For example, there are "fictitious" forces felt (instantaneously and locally) in the accelerating frame but not in the IIRF (noting that these "fictitious" forces are real and have real physical effects represented for instance by the pull and push felt in the accelerating frame but not in the IIRF).

5. Give an example of a limitation of the equivalence principle related to the local application of special relativity in the curved spacetime of general relativity.
   Answer: Let have a special relativistic law that depends on the second order partial derivative of a tensor (say $A_{\alpha,\beta\gamma}$) and we generalize this law (to be applied in the curved spacetime of general relativity) by changing the partial derivative to covariant derivative (i.e. $A_{\alpha;\beta\gamma}$) to account for the curvature of spacetime. Now, when we apply this law locally in the curved spacetime of general relativity we cannot simply restore the original form (i.e. $A_{\alpha,\beta\gamma}$) because this will annul the effect of the curvature of spacetime entirely. The reason is that although we can make the Christoffel symbols in the second order covariant derivative vanish at the pole of the local inertial frame we cannot force the vanishing of their derivatives (refer to § 2.10). In brief, converting the second order covariant derivative to a second order partial derivative means annulling all the effects of spacetime curvature as if we are in a globally flat spacetime and the generalization is simply void.
   Note: this question has obvious link to the previous question.

10.1.2 Necessity of Metaphysical Elements

1. Outline the problem of dark matter.
   Answer: We refer the reader to the following quote which outlines eloquently the problem of dark matter:[230] "All relativistic gravitational theories mentioned so far have a Newtonian limit, a tacit requirement of candidate relativistic gravitational theories until very recently. Now, if the correct gravitational theory is general relativity or any of its traditional imitations, then Newtonian theory should satisfactorily describe galaxies and clusters of galaxies, astrophysical systems involving small velocities and weak potentials. But there is mounting observational evidence that this can be the case only if galaxies and clusters of galaxies are postulated to contain large amounts of dark matter. Thus far this dark matter has not been detected independently of the preceding argument".
2. Argue against the existence of dark matter.
   Answer: We first note that we distinguish between two types of dark matter:
   • Ordinary matter that is "dark" because it cannot be observed directly such as dim stars and cold interstellar matter.
   • Non-ordinary matter (i.e. a form of matter that we are not familiar with from our terrestrial and celestial observations) such as the alleged non-baryonic matter hypothesized in some cosmological

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[229] In fact, this can be seen as an implicit reference to the *Principle of General Covariance* (which may be seen to represent the strong equivalence principle in its most rational form).

[230] This quote is taken from an essay by Jacob Bekenstein which is posted on the Internet.

models.

Primarily, we reject only the existence of the second type on the basis of being a metaphysical entity. However, even the existence of the first type in any particular circumstance should be accepted only if it is based on a credible physical evidence that is independent of our theories and models (i.e. it is not used as an *ad hoc* fix for our failing theories and models). Accordingly, even the first type of dark matter should be regarded a metaphysical entity if there is absolutely no evidence for its existence other than being a convenient fix for our theories.

Apart from being a metaphysical entity and hence it should be rejected on this basis, we can argue against the existence of dark matter of the second type by the fact that if we have to accept it as a scientific paradigm then we should also accept the possibility of the existence of different set of physical laws in other parts of space and time (or spacetime). In fact, if we accept this possibility then large parts of our models and results in astronomy and cosmology could collapse because almost all these models and results depend on the implicit presumption that the physical laws are essentially invariant in space and time[231] although there may be variation in some "constants" or parameters or dominance of certain physical phenomena or regimes in certain locations and eras (as well as the possibility of the existence of some yet-undiscovered laws at the very beginning of the Universe for instance). In fact, there is no fundamental objection to the acceptance of metaphysical entities (like dark matter of the second type or dark energy) or the acceptance of non-invariance of the physical laws in space and time but this requires a fundamental change in the epistemological principles and foundations of science (and no one seems to be willing to follow this route which should change the nature of science).

3. It may be claimed that the existence of dark energy can be justified by the "observed" expansion of the Universe and hence dark energy has physical justification and evidence. How do you address this claim?

   Answer: We note the following:

   • A physical expansion (assuming it really exists) should be explained by physical entities. If this alleged expansion cannot be explained physically then we should address this by checking and assessing our models and theories rather than by resorting to metaphysics.

   • The expansion is not really observed but it is inferred (possibly wrongly) from our models (plus some observations). So, if this alleged expansion requires metaphysical entities for justification then we should question and examine this expansion which led to the necessity of metaphysics.

   • If we assume that this expansion and the Big Bang (which this alleged expansion is based on) are verified facts, then we do not need science in this situation because we have already crossed the border between physics and metaphysics since the creation of the Universe is a metaphysical premise and hence if we are in the domain of metaphysics then we do not need dark energy or energy (or indeed even physics) to explain the expansion as we do not (and in fact can not) explain the creation itself by our physics.[232] In brief, the metaphysical premise of creation (which led to this expansion) allows the resort to any sort of metaphysical premises with no need (and indeed with no permission) to justify our metaphysics by our physics. To put it in another way, when we are dealing with issues at the scale and level of creation (and subsequent expansion) then physics does not and should not apply not only because it is not needed but also because it has no subject and hence using physics at this scale and level is fundamentally wrong. So, we should not contaminate our metaphysics with physics or our physics with metaphysics.

   • Even if we assume that the expansion is independent of creation, we think that at this scale and level of cosmological events we should not use our physics because physics is about what is inside this world and not about the world itself and hence it does not have the mandate and legitimacy to go beyond this and look into issues at the scale of the world itself by using the "local" laws of physics (such as the necessity of energy to cause expansion) to explain the "global" or cosmological expansion.

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[231] This invariance is different from the invariance across coordinate systems and frames of reference.

[232] For example, if the creation is explained by the existence of a supernatural entity or deity, then the expansion can also be legitimately explained by the existence of this entity or deity.

10.1.3 Creation Theory

1. Discuss the issue of conservation of energy in general relativity and its applications.
   Answer: In fact, this is one of the gray and hazy areas in the literature of general relativity where we can find conflicting opinions and suggestions. For example, we can find assertions of the conservation of energy as well as opposite assertions. There are also claims of the conservation of energy at local but not global level (which we generally accept although our justification and interpretation may not be the same). In fact, deep inspection of the literature reveals that this issue is not clear cut even among the experts in this field and hence this issue is generally avoided and not discussed in detail due apparently to suspicions and lack of clarity. This avoidance may also be caused in some cases by a sense of embarrassment since the conservation of energy is one of the most fundamental principles in physics and hence it seems absurd to suggest the violation of this principle or accept a theory that does not respect it.

   Apart from the lack of significance of energy and energy conservation at global level (see for example § 7.8 and § 7.9), the conservation of energy in our view is not totally respected in general relativity (or at least this is the case in some of its applications, such as cosmology, which should suggest that the theory can accommodate the violation of this principle). An example of the violation of energy conservation is the cosmological expansion of the Cosmic Microwave Background Radiation which should violate the conservation of energy according to the Planck's law.[233] Another example is the dark energy where the expansion of the Universe with the presumed constancy of the density of dark energy (at least according to some cosmological models) should lead to the violation of energy conservation since dark energy should be created to keep its density constant as the Universe expands.

   Note: general relativity (or its applications) may also violate other conservation principles (or alleged conservation principles) such as the conservation of baryon number. The conservation of momentum (due to the relativity of length, direction as well as time and energy) may also be questioned. There are also suggestions of other potential violations (e.g. violation of the second and third laws of thermodynamics in black holes). Also, some features of the Kerr solution seem to violate causality. These issues are investigated rather tersely in the literature (which the interested reader should refer to). However, regardless of the outcome of these violations or alleged violations, these issues should put question marks on many aspects of general relativity and should call for more careful inspection to the theory and its applications and implications. It is quite possible that some of these gray areas indicate inconsistencies and defects in the theory itself.
2. The necessity of the conservation of energy principle in physical theories may be challenged. What is your argument against this challenge?
   Answer: We may argue against this challenge by the following:

   • Epistemological argument: the breakdown of this principle implies creation which is non-physical (see § 1.9).

   • Physical argument: since this principle holds true in all our observations then any theory that violates this principle should be rejected because such a theory is not consistent with our observations. In fact, such a theory requires the violation of this principle for its own consistency and not for the consistency with observations and hence the theory cannot be a legitimate basis for violating this principle. Yes, if we observed a violation to this principle in some cases then a theory can accommodate this violation in its theoretical framework even beyond these cases because the observed violation justifies the assumption that this principle is not general and hence in principle it can be violated even in other cases. In fact, the necessity of the conservation of energy principle in physical theories can be considered a generalization of the correspondence principle (see § 1.8.3) because in essence any new theory should be consistent with our actual observations prior to the emergence of that theory (regardless of any other previous theory). This generalization is legitimate to keep our theories connected to physical reality. This is also inline with our demand that physical theories should follow

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[233] This example may be disputed by the proposition that the reduction in energy by the expansion of wavelength is compensated by the expansion of volume, but this can also be disputed. We should note that this example is not specific to general relativity or any particular cosmological model since it is based on the cosmological expansion itself.

(rather than be followed by) observations. Yes, new theories may contemplate about things that are not observed but these theories cannot be accepted (as it is the case with general relativity) before the emergence of some observational evidence or indication in support of their validity; otherwise physics will be chaotic since anyone can cast his illusions into a "legitimate theory". In brief, any legitimate new theory should comply with our verified laws and principles (which include the conservation of energy) until we find observational evidence (or at least indication) that these laws and principles are invalid or not general. So far, no theory that violates the conservation of energy principle based its violation on observation; instead these theories base their violation on the validity of the theory itself because the correctness and consistency of the theory requires this violation. In fact, this should include even quantum mechanics where the violation is represented by the uncertainty principle and its consequences.

3. Assess the assertion that the breakdown of the conservation of energy at global level in general relativity can be seen as a logical consequence of the breakdown of global time since time in the spacetime of general relativity is well defined only locally.
   Answer: This may lead to the conclusion that general relativity cannot be classified as a creation theory from this perspective since energy will be a relative and local quantity (i.e. frame and coordinate dependent) and hence its conservation (i.e. in absolute sense and at global level) is meaningless. However, as we saw earlier there are instances (or alleged instances) of the violation of energy conservation in an absolute sense (i.e. regardless of the global insignificance of time and energy) and hence general relativity can be blamed for being a creation theory (or at least accepting and accommodating creation).
   Note: to be more clear about the difference between the two types of violation of energy conservation in general relativity, we note that the violation (or rather physical insignificance) in one type is based on the insignificance of time and energy at global level (and hence energy at one event or from one frame cannot be contrasted with energy at another event or from another frame in an absolute and global sense), while the violation in the other type is based on creation (or annihilation) of energy.

10.1.4 Dependence on Special Relativity

1. Give an example of the challenges to special relativity that potentially threaten general relativity.
   Answer: A serious challenge from this perspective is the status of the speed of light (which is constant, invariant and ultimate speed according to the second postulate of special relativity) since this status is challenged by a number of observations and experiments as well as by quantum entanglement and non-locality. In fact, the latter seems to be the most serious threat to the second postulate of special relativity (and hence to special relativity itself and possibly even to Lorentz mechanics) at the moment since many elaborate experiments seem to confirm quantum entanglement and non-locality. Also see § 10.1.13 and § 10.1.22.

10.1.5 Triviality of General Invariance

1. The principle of invariance is no more than an epistemological convention with minor scientific content and significance. Discuss this proposition.
   Answer: It should be obvious that the invariance of physical laws is a very useful principle. However, this principle is no more than an epistemological demand, i.e. there is no fundamental physical demand for this invariance. In fact, the necessity (or rather usefulness) of the invariance principle originates from the quest of better adaptation with our physical environment rather than from an intrinsic physical necessity. For example, let have two different frames of reference and we formulated the physical laws in a certain form (say form A) in one of these frames (say frame 1) and formulated our laws in a different form (say form B) in the other frame (say frame 2). As long as we can describe and predict the physical phenomena using form A when we are in frame 1 and using form B when we are in frame 2, then these laws are fully functioning and they achieve our objectives of adaptation with our physical environment.

2. Distinguish between the invariance of laws in essence and the invariance of laws in form.
   Answer: The invariance of laws in essence originates from the principles of reality and truth which demand that the actual content of the law should be independent of the observer and frame of reference (as long as the change of the observer and frame does not introduce real physical change). For example, changing the orientation of the axes of the frame does not introduce real physical change (assuming the isotropy of space) but shifting from inertial frame to accelerating frame should introduce real physical change (assuming the existence of absolute frame). Accordingly, the invariance of laws in essence is a physical and logical demand that cannot be violated. On the other hand, the invariance of laws in form is about the mathematical formulation of the laws in a certain shape and form and hence it can be regarded as an "aesthetic" issue whose necessity stems from pragmatic and purely epistemological objectives.
   Note: in fact, the invariance of laws in essence stems from the principles of reality and truth which are also epistemological in nature. However, they are more fundamental since they have physical roots and can be related to the issue of causality (i.e. real change in the law cannot be caused by a conventional change in coordinates and frames).
3. Clarify the assertion about the triviality of the alleged general invariance (or general covariance) in general relativity.
   Answer: No specific transformations have been suggested to implement the alleged general invariance of physical laws. The demand of being cast in tensor form (and hence they are supposedly subject to the most general type of transformations) is not sufficient. Moreover, it should be shown in reality and in detail that this general invariance actually takes place when applied to actual physical laws (similar to what is done in special relativity where the invariance of physical laws across inertial frames under the well defined Lorentz transformations is shown in detail). Furthermore, no physical evidence is given for the validity of the principle of general covariance and hence it is no more than an empty claim. In fact, even its logical validity and sensibility are questionable due to the lack of convincing rationale behind this alleged general covariance.
4. Justify the triviality of the *Principle of General Covariance* (see Problems of § 1.8.2).
   Answer: The main aspect of triviality is that this principle is not substantiated by physical evidence[234] and hence it is an empty claim, although some of the earlier criticisms and triviality aspects may not apply to it. In fact, even the logical sensibility of the *Principle of General Covariance* can be challenged because the validity of a law in a local inertial frame does not imply the validity of its transformed form in another frame that represents different physical setting due to the real physical difference between the two frames, i.e. a freely falling frame and a stationary gravitational frame for instance are physically different rather than conventionally different. In other words, the difference between the two frames is not conventional but it is essential due to real physical difference and hence the law in one frame should not be obtainable from the law in the other frame by just applying a formal tensorial transformation that supposedly keep the form invariance aspect without affecting the essence (because if the physical setting is different then even the essence should be different).[235] Yes, if we accept the strong equivalence principle (which is flawed or at least dubious) then we may accept this *Principle of General Covariance* (although even then we may question some aspects of its application).

10.1.6 Interpretation of Coordinates

1. What is the root of the idea that coordinates are labels with no metrical value?
   Answer: The root is modeling the physical spacetime as curved mathematical space. In mathematics

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[234] We may even claim that there is evidence against it as demonstrated earlier in § 10.1.1. We should also note that the invariance is a necessary but not sufficient condition for a rule to be a law of physics (see § 1.8.1) and hence the formal form-invariance is not sufficient to establish the actual (or physical) invariance which is what is required for the validity of the *Principle of General Covariance*. We should also note that the necessity of the existence of a valid transformation in general is not evident and hence the *Principle of General Covariance* may not be applicable in some cases.

[235] In fact, the logical sensibility of this principle should depend on the claim that the formal transformation accounts for the effect of the real physical difference between the two frames and settings, and this claim is not obvious (in fact there are indications and instances of its falsehood).

we have no problem with the use of curved space because we are dealing entirely with an abstract world and hence we do not have a corresponding physical world to model and worry about. Also, in physics we have no problem in using ordinary flat space in modeling physical phenomena because the mathematical coordinates and the physical coordinates of such space are identical since they correspond to each other with identity metric transformations (and hence the metric is effectively redundant with all its geometric cost). But when we use curved space in a physical theory the abstract mathematical coordinates do not correspond identically to the real physical coordinates and hence we need metrical relations to link the two types of coordinates. The whole problem will be solved effortlessly if we abandon the general relativistic paradigm of curved spacetime which is at the root of this problem (and indeed at the root of many other problems and difficulties in formulating, managing, understanding, etc.). Instead, we should use the paradigm of flat spacetime in which all physical phenomena (including gravity and all its consequences) take place.[236] This will restore the beauty and simplicity of classical physics (or rather classical view and philosophy) and protect physics from the many traps and complexities of this curved view. This will also save a huge amount of effort and resources and protect us from many sources of confusion and error. This requires of course a fundamental change to classical physics which should consider the inclusion and accommodation of essential modern physical theories like Lorentz mechanics and quantum mechanics.[237] Moreover, the merge of these theories should require considerable amount of reformulation and reinterpretation. This may be the right way to a real "General Theory" instead of a theory that singles out gravity (for personal reasons that originate from the historical development as outlined earlier in § 1.2) and makes it the center of everything where this gravity (and its theory) distorts the spacetime (which should be a container inside which all the physical phenomena take place) and dictates on the other physical phenomena (and their theories) to adapt with this distortion and mess. In simple words, gravity is a physical phenomenon like any other physical phenomenon that should take place in spacetime rather than shaping the spacetime and being embedded in its structure, and accordingly if other phenomena can take place in flat spacetime then gravity can also take place in flat spacetime.[238] So, the central general relativistic paradigm of curved spacetime is questionable as a wise and right choice.

2. List some of the problems caused by the interpretation of coordinate variables as labels with no metrical significance (which originates from employing the paradigm of curved spacetime).
   Answer: For example:
   • Data representing physical systems or extracted from experiment and observation cannot be used because they are metrical while the required data in the formulae (i.e. the non-metrical coordinate variables) are not available.
   • Many coordinate-related quantities (e.g. length, time, energy, etc.) lose their global physical significance. This does not only impose conceptual difficulties but even practical difficulties. For example, in the absence of global physical significance of length what is the meaning of the physical (or metrical) radius of the Earth when we use it in a formula that is supposed to require the distance between the center of the Earth and its surface (where this distance should have a global significance since it extends over different patches of spacetime with different metric properties)?
   • Common physical quantities lose their unique and distinctive meaning and hence they will have different meanings and definitions which depend on the context, case and circumstance. For example, in the astronomical and cosmological applications of general relativity we do not have a unique definition for (proper) distance and hence we have for example ruler distance, luminosity distance, radar distance and parallax distance.[239] This also imposes practical (as well as conceptual) difficulties

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[236] In fact, there are numerous non-classical gravitational theories that are based on flat spacetime.

[237] By the way, both these well tested theories (despite their limitations) do not require curved space or spacetime.

[238] We should similarly say: "and if the spacetime is actually curved (i.e. intrinsically and not because of the presence of matter) then all phenomena (whether gravitational or not) should be modeled in this curved spacetime". However, we believe that flat spacetime is sufficient to model all physical phenomena and formulate all physical theories.

[239] The reader should note that these different definitions do not differ only conventionally but they also differ physically and in reality since the observations based on these definitions differ. For example, the measured values of "distance" depends on the technique used in the measurement and on which definition the employed technique is based (and hence ruler distance is physically different from radar distance). This could lead to many inconsistencies in the application of

when we apply the formulation, e.g. why should we choose one type of a distance but not another and what is the physical significance and justification of the obtained results from our choice? In fact, in some cases this could lead to chaos in the application of the general relativistic formulation and arbitrariness of the results and hence it can cast a shadow on their physical significance. It can also damage the value of any claimed evidence in support of this formulation since we do not have a result with a unique and definite significance. In fact, the mere vagueness caused by this multiple-significance of physical quantities (even if we assume internal logical consistency) is a major source of confusion in understanding, interpretation, application, generalization, correlation, etc. and this is not a trivial problem since it inevitably leads to many mistakes, complications, controversies, etc. which do not serve the development and progress of science.

- Ambiguity of the characteristic speed of light c where distance and time (which enter in the definition of c) are not well defined (at least globally). In fact, this is an instance (and indeed a very important and prominent instance) of the aforementioned problem of losing global physical significance, but it is singled out due to the special status of c in the relativity theories. It may also be related to the problem of multiple meanings (where the distance traversed by light should have a special physical meaning).

3. It may be claimed that "As long as the coordinates are not measurable quantities then the actually-measurable physical quantities cannot be correlated in reality to these coordinates. In other words, the actual link between coordinate quantities and physical quantities is missing in reality because this link is theoretical rather than practical". Discuss this claim.
Answer: This claim can be easily challenged by the existence of metrical relations that link the measurable physical quantities with the non-measurable coordinate quantities, and hence even the non-measurable quantities can be obtained in reality. However, this challenge may address the problem theoretically but not practically because the measurable and non-measurable quantities are usually combined in the metrical relations in such a way that makes obtaining the non-measurable quantities mathematically intractable (as well as being physically inaccessible). Accordingly, the above claim should have at least some practical credibility.

10.1.7 Strong Dependency of Physical Results on Coordinate System

1. Try to find a link between the issue in this subsection (i.e. strong dependency on coordinates) and the issue in the previous subsection (i.e. interpretation of coordinates).
Answer: The issue in this subsection provides more clarification about the issue in the previous subsection (with regard to the nature and role of coordinates) because in general relativistic situations (or in some of them at least) the claim that the coordinates are just labels is not true since they are essential in determining the physics of the problem (or at least suggesting the physical implications and interpreting the physical results) and hence they actually play more important role than the role of mere labels even if we accept that they have no metrical significance. In fact, both issues arise from the use of the geometric approach of curved spacetime to model physical phenomena.
2. Give an example from the literature of general relativity about the strong dependency of the physical results on the type of coordinates used in formulating and solving physical problems. Also, discuss possible challenges and defence to this criticism.
Answer: An example of the strong dependency is the differences in the physical nature of the Schwarzschild solutions obtained from using the Schwarzschild coordinates, the advanced Eddington-Finkelstein coordinates, the retarded Eddington-Finkelstein coordinates and the Kruskal-Szekeres coordinates (where these differences are seen prominently in the investigation of the physics of black holes). Although, these differences are justified in the literature by being features and artifacts of the coordinates or/and by being representatives of different parts and features of the manifold and its geometry the reality is that the ability and power of the coordinates in general relativity to lead to (or at least to suggest) such different physical results (even if we accept these justifications) makes the physical interpretation of the physical results that we obtain from general relativity susceptible to sus-

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any theory (like general relativity) that involves these different definitions.

picion and mistrust since we may be misled by these artifacts and limitations. In fact, this is not just a hypothetical scenario but it happened in the past where the singularity at $r = R_S$ in the Schwarzschild solution that is based on the Schwarzschild coordinates was historically interpreted wrongly (with inevitable conclusion of wrong results and implications) and the "misconception" persisted for a rather long time before "discovering the truth". In our view, the problem of strong dependency of physical results on the type of coordinates is just another demonstration of the danger of over-mathematization of physics and using geometric paradigms (especially curved spacetime with its dual coordinate and physical variables) to model physical phenomena which is the fundamental methodology of general relativity.

10.1.8 Local Relativity

1. Can you give an example of a physical quantity in the spacetime of general relativity that is absolute and global? What is the significance and implication of this?
   Answer: Mass should be an absolute and global physical quantity because it is not coordinate dependent (unlike length or time or energy for instance). Accordingly, local relativity in general relativity is not general, i.e. we have some local and relative quantities (like time) and some global and absolute quantities (like mass). The implication of this is that we could face an incompatibility problem in some circumstances where a local quantity is required to match and correspond to a global quantity. For example, according to the principle of mass-energy equivalence a global quantity (i.e. mass) is supposed to equate to a local quantity (i.e. energy) and this should lead to an incompatibility problem which should have an impact for instance on the conservation of mass-energy and its global/local significance and interpretation. Yes, if some types of energy are local while other types are global then we may escape this dilemma. However, even if this is sensible it may not be able to address all the problems and challenges.

10.1.9 Nonsensical Consequences and Predictions

1. Give some examples of some nonsensical general relativistic results that violate certain physical rules and concepts.
   Answer: For example:
   • Some general relativistic solutions imply that one can travel back in time and meet himself in the past (which can lead to violations of causality).
   • Things like negative mass and negative energy (which are found in the literature of general relativity as possible physical entities implied by legitimate general relativistic formulations) violate the intuitive and sensible meaning of mass and energy and can lead to nonsensical and contradictory results.
   • There are also many absurd results and nonsensical consequences related to the cosmological constant (some of these are indicated in § 3.3 such as negative pressure, negative gravity, anti-gravity, and positive pressure slowing down expansion).
2. It may be claimed that general relativity is not different from classical physics in having nonsensical consequences and predictions because even in classical physics we may obtain nonsensical results (in the form of non-physical solutions such as imaginary roots) from legitimate classical formulations. Address this challenge.
   Answer: We note the following:
   • Such "nonsensical" results are rather rare in classical physics, but they are very common in general relativity.
   • "Nonsensical" results in classical physics are usually associated with sensible results and hence we are able to choose the sensible and discard the nonsensical without confusion or hesitation, but in general relativity we commonly find nonsensical results with no alternative sensible results and hence the choice of nonsensical may seem inevitable.
   • The "nonsensical" results of classical physics are simple, naive and innocent and hence no one believes in their reality, while the nonsensical results of general relativity are elaborate, sophisticated and malicious and hence many believe (or can believe) in their reality. This is highlighted by the fact that the

nonsensical relativistic results are given serious attention and consideration in the literature of general relativity, while the "nonsensical" classical results are easily dumped (usually with a brief comment or justification such as "non-physical solution"). In fact, this is ultimately based on the realistic nature and intuitive methodology of classical physics which keeps classical physicists close to reality, in contrast with the fanciful nature and counter-intuitive methodology of general relativity which facilitates the detachment from physical reality and makes such nonsensical results look acceptable and natural.

10.1.10 Incompatibility with Quantum Mechanics

10.1.11 Gaps, Ambiguities and Question Marks

1. Discuss and assess the issue of whether the gravitational field itself contributes to the source of gravity (as represented primarily by the distribution of matter and energy) or not and whether gravitational forms of energy (such as gravitational waves) make such a contribution or not.
   Answer: This is a complicated and rather controversial issue. It is commonly classified as an issue of non-linearity in the formulation of the theory although it could be classified (at least in some of its aspects and instances) as a problem of circularity. From a practical perspective this issue is trivial in most cases although it can be important in extreme gravitational fields (e.g. in the neighborhood of stellar black holes). However, from a theoretical perspective it is of immense importance because it could put serious question marks on fundamental paradigms of general relativity such as sourcing the gravity by energy. In fact, the lack of global significance of energy in general relativity could aggravate the challenge to general relativity from this perspective. We believe that most of these troublesome aspects originate from the principal methodology of general relativity of singling out the gravity as a curvature in the spacetime instead of treating gravity as an ordinary physical phenomenon (like electromagnetism for example) that takes place in the spacetime.
2. According to the *Principle of General Covariance* the effect of gravity on a physical system can be assessed by the procedure described in the Problems of § 1.8.2 (also see exercise 3 of § 3.5). Accordingly, the problem of stationary local frame in a gravitational field (or non freely falling gravitational frame to be more general) is not a gap or ambiguity or questionable issue since the effect of gravity can be accounted for by this procedure. How do you respond to this?
   Answer: This apparently fills this gap (as well as other related gaps) since the effect of gravity in the stationary frame will be accounted for by the transformation following the generalization of the law. In fact, by the equivalence principle even the effect of acceleration can be assessed by transformation either directly from the special relativistic frame or from the equivalent gravitational frame. However, we note the following:
   • There is no physical evidence for the validity of this procedure (whose essence is the strong equivalence principle which is questionable) in accounting for the effect of gravity.
   • There is no logical necessity of the claimed equivalence between the two systems due to the fact that the applicability of a mathematically valid general transformation does not imply the applicability of a physically valid transformation that accounts for all the involved physical effects in the two systems and the incorporation of the effect of gravity in particular in this transformation.
   • At least this is an ambiguous and questionable aspect in the theory and its logical application since the rationale of the equivalence in this procedure is not clear.
   • We may also challenge this procedure by the many ambiguities and question marks (as well as restrictions) related to the strong equivalence principle which is at the foundation of this procedure (e.g. its restriction by locality as well as limitation on its validity and generality).
3. Discuss briefly the gaps (and potential gaps) in the investigation of the principles of invariance and equivalence (and their combination as represented by the *Principle of General Covariance*) in the literature of general relativity.
   Answer: In our view, there are many gaps and hazy issues in this investigation and this leads to many inconsistencies and controversies. Some examples of these gaps are:
   • Lack of distinction between the invariance of the essence of the law and the invariance of its form.
   • Lack of characterization of the nature of the difference between reference frames and if this difference

is conventional (and hence it should not have real physical impact on the transformed laws) or physical (and hence it should have real physical impact on the transformed laws).
- Lack of clarity about the nature of these principles and the relation between them.
- Lack of clarity about the relation of these principles with the principle of relativity.

10.1.12 Over-Mathematization of Physics

10.1.13 Violation of Sacred Rules

1. Give an example of a general relativistic violation (or potential violation) of a sacred rule from special relativity, and assess in detail the significance of this violation and its impact on special relativity as well as on general relativity.
   Answer: The obvious and most prominent example is the violation of the constancy of the speed of light (which is supposed to be one of the sacred rules at least in special relativity) where in general relativity the constancy is restricted by locality. Although this may seem consistent it essentially violates this rule in reality and defeats the purpose of keeping it. If the legitimacy of this rule originates from Lorentz mechanics (as discussed in B4) then this may indicate a limitation in the existing Lorentz mechanics or in its domain of applicability. So, instead of keeping this rule formally while violating it in reality we may need to look for a different formulation or interpretation or approach that can discard this rule altogether (although it can be still valid in special cases) and free ourselves from this obligation. The effect of this is that in this case we can find more logical and intuitive explanations to known physical phenomena in which the speed of light is not constant (e.g. superluminal speeds) than the current explanations which struggle to keep this rule alive. Although this may not be a matter of right and wrong, it is a major source of confusion, struggle and controversy in modern physics about these phenomena and how they should be explained and justified.
   In fact, this problematic issue (like several other problematic issues that we discussed earlier) is closely related to the general relativistic paradigm of curved spacetime (which originates from the special status of gravity) because if our spacetime (which includes gravity and lifts the restriction of inertiality by including non-inertial frames) is flat then the constancy of the speed of light is obviously violated because we do not have any distinction between local and global. So, all we need to explain and justify is why the speed of light in the Lorentz spacetime (or Minkowski space) should be constant. In other words, the constancy of the speed of light is restricted to the Lorentz spacetime and it can be justified by a limitation of the formalism or/and the inertiality of frames (which is based on the existence of absolute frame) and hence when we reformulate and generalize the theory or/and lift the inertiality of frame we could have variation in the speed of light. In fact, we think this should be addressed fundamentally by changing the existing formalism of Lorentz mechanics so that it can accommodate all forms of physical phenomena (including gravity) and all types of frame (including accelerating and rotating frames).
   In brief, many of the confusions and inconsistencies in modern physics seem to originate from the general relativistic paradigm of curved spacetime (and the special status of gravity) and hence they can be avoided by changing this to the more natural and intuitive paradigm of classical physics where space-time is flat and it is represented and regulated by the existence of a single absolute frame of reference. In fact, this classical paradigm is logically and epistemologically consistent with the essence and formalism of Lorentz mechanics (as demonstrated in B4) and hence our problem is restricted to general relativity. If we adopt this classical paradigm (which can also be seen as Lorentzian paradigm) then we should be able to produce a more simple physics and avoid many traps, difficulties and paradoxes of modern physics. So, even if modern physics (and general relativity in particular which represents the lifted face of modern physics) is essentially correct, it is unnecessarily complicated and distorted and we are paying very high price for this in terms of effort, progress, resources, etc.
   We should finally remark that the generalization of Lorentz mechanics to all physical phenomena including gravity (where the spacetime should remain flat) may be objected by the non-invariance of gravity under Lorentz transformations which leads to the alleged necessity of employing the paradigm of curved spacetime and using the most general tensor-type transformations. The essence of this ob-

jection is that Lorentz transformations are valid transformations for flat spacetime excluding gravity but they are not valid transformations for spacetime with gravity because such a spacetime is curved by gravity. However, the problem of the presumed limited validity of Lorentz transformations (if they are really so) does not necessarily lie in the nature of the spacetime (i.e. being flat) but possibly in the nature of the transformations themselves, i.e. Lorentz transformations could be a special type of a more general type of transformations that can transform all the laws (including gravity) invariantly in flat spacetime.[240] Accordingly, the Lorentz transformations are like the Galilean transformations which are a special case of the Lorentz transformations.

In fact, if Lorentz transformations proved to be invalid for transforming gravity invariantly in flat spacetime then they should be discarded as we discarded the Galilean transformations due to their failure to transform electromagnetism invariantly although they transform mechanics invariantly. In other words, if we accept certain transformations because they transform some physical laws invariantly then we should accept the Galilean transformations because they transform classical mechanics invariantly, and if we reject certain transformations because they transform only some (not all) physical laws invariantly then we should reject Lorentz transformations because they fail to transform gravity invariantly. So, instead of replacing the flat spacetime we may need to replace the transformations while keeping the spacetime. In fact, we may even need to reformulate gravity (with or without employing a new set of transformations) to become invariant under the Lorentz transformations or any new set of transformations (as we reformulated classical mechanics to make it invariant under the Lorentz transformations).

Anyway, Lorentz transformations have another limitation in their validity which is their restriction to inertial frames. So, in our view a proper "General Theory" should first extend Lorentz mechanics to include gravity in a flat spacetime within the inertiality restriction, and it should second lift the inertiality restriction and hence generalize the observer, i.e. we need a generalization of the type of physical laws (i.e. subject) to include gravity and the type of frame (i.e. domain of validity) to include accelerating frames. If so, then we will have a theory that transforms all physical laws (including gravity) invariantly for all types of frame (including non-inertial) in a flat spacetime. We think this is a more rational approach than the general relativistic approach which distorts the spacetime by gravity, then tries to address the generalization to accelerating frames by the equivalence principle and finally tries to address the general invariance by arbitrarily imposing an elusive demand of casting the laws in tensor form.

2. Give another example of a violation (or potential violation) of sacred rules by general relativity.
   Answer: It is the violation of the conservation principles (and the conservation of energy in particular), as discussed earlier (see for example § 10.1.3 and exercise 1 of that section in particular where several other examples were given in its note).
3. The call for formulating the physical theories (including gravity) using flat spacetime may be challenged by the possibility that the physical spacetime could be inherently curved. Discuss and assess this issue briefly.
   Answer: This issue has been discussed previously. In brief:
   • The curvature of the physical spacetime (if the curvature really exists) does not justify distinguishing gravity from other physical phenomena by making gravity a curvature of spacetime rather than an ordinary physical phenomenon that occurs in spacetime. The cause of the spacetime curvature could be the nature of the spacetime itself (regardless of any physical factor like gravity) or some other reason and hence correlating gravity specifically to this curvature requires justification and evidence. In brief, "the curvature of spacetime" and "attributing this curvature to gravity specifically" are two different and independent issues.
   • Based on the previous point, if the physical spacetime is really curved inherently then this requires formulating all the physical theories (not only gravity) in accord with this curvature. Hence, gravity will not enjoy a special status in its relation to the curvature of spacetime (i.e. it will not be represented as a curvature of this spacetime but as a physical field in this spacetime).

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[240] The presumed limited validity may also be caused by the particular formulation of gravity (as explained earlier and will be indicated later).

- The available physical observations indicate that the physical spacetime is actually flat. At least, this is a very good approximation and hence for all practical purposes the physical spacetime can be treated as flat.
- The effect of any presumed inherent curvature in the physical spacetime can be incorporated in the physical theories as a physical element (rather than as a geometric element) while keeping the theoretical spacetime flat. What should facilitate this is the tininess of any such inherent curvature (as indicated in the previous point) because the effect of curvature can be treated as a physical field imposed on a flat-spacetime background.
- Regarding general relativity, all the physical effects that are explained in this theory by the alleged curvature of spacetime can be explained by physical (rather than by geometric) factors, i.e. by taking gravity as a physical field in the spacetime rather than a structural factor of the spacetime. The effect of any presumed inherent curvature of spacetime can then be modeled as an ambient background physical field that superposes on the gravity field.

10.1.14 Circularity

10.1.15 Using Einstein Tensor to Represent Curvature

1. Summarize the essence of the issue discussed in this subsection.
   Answer: According to the central paradigm of general relativity, gravity is correlated to curvature. Now, we can ask: why the curvature is partly represented, and why this representation should be in this particular form?
2. What is the implication of using the Einstein tensor (rather than the Riemann-Christoffel curvature tensor) to describe gravity?
   Answer: The implication is that gravity is linked to a certain part of the spacetime curvature (i.e. the part represented by the Ricci curvature tensor) but not to all parts of the spacetime curvature (since the part of the curvature that is represented by the Riemann-Christoffel curvature tensor excluding the part represented by the Ricci curvature tensor is not correlated to the energy-momentum tensor and hence it does not represent gravity).
3. Propose a challenge to the following statement: using the Einstein tensor to represent and describe gravity in general relativity is arbitrary and incomplete representation of gravity since the rank-2 Einstein tensor captures only part of the spacetime curvature due to the fact that this curvature is fully represented by the rank-4 Riemann-Christoffel curvature tensor.
   Answer: This statement may be challenged by the physical evidence in support of the theory since the theory (regardless of any theoretical arbitrariness) should be correct if it is endorsed by experimental and observational evidence. However, even if we accept all the claimed evidence this will not solve the problem or address this issue because all the available evidence is partial and hence it cannot rule out possible failure of the theory in some cases and circumstances where the effect of the missing part of the curvature is dominant and tangible.[241] Moreover, even if the theory proved to be empirically correct, this does not address the problem of inconsistency in its theoretical structure where gravity is attributed to the curvature of spacetime whereas curvature is arbitrarily defined and not totally represented.
4. A justification for the seemingly arbitrary choice of the Einstein tensor $G^{\mu\nu}$ to represent the curvature of spacetime is that it is the only rank-2 tensor that is second order in the derivatives of the metric for which the divergence vanishes (i.e. $G^{\mu\nu}_{\ ;\nu} = 0$). Is this justification sufficient to address the issue of arbitrariness and partial representation of curvature?
   Answer: This obviously does not address the issue of the arbitrariness of representing the curvature partially and specifically by this tensor. In fact, it does not address even the issue of representing the gravitationally-significant part of the curvature by the Einstein tensor. We should remark that the justification by the conservation of energy-momentum or 4-momentum (since $G^{\mu\nu}_{\ ;\nu} = 0$ leads to

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[241] For example, the need for dark matter or dark energy may be caused by this incomplete representation of gravity by curvature.

$T^{\mu\nu}_{;\nu} = 0$ or the other way around) is also not sufficient especially if we note that this conservation may not be valid (at least because of the lack of global significance); moreover the correlation between $G^{\mu\nu}$ and $T^{\mu\nu}$ is based on the validity of the Field Equation (whose validity is supposed to be under the inspection and criticism).

5. Comment on the following quote from Rindler (see References): "The choice of $G_{\mu\nu}$ for the LHS of the field equations may at first seem somewhat arbitrary. But it can be shown [cf. D. Lovelock, J. Math. Phys. 13, 874 (1972)] that $G_{\mu\nu}$ is the only tensorial and divergence-free function of the $g_{\mu\nu}$ and at most their first and second partial derivatives, that, when put on the LHS of the field equations, allows Minkowski space as a solution in the absence of sources".

   Answer: Even if we accept this claim it does not justify the choice of the Einstein tensor from the perspective of partially representing the curvature of spacetime. In fact, other excuses can be found in the literature for the choice of the Einstein tensor to represent the curvature of spacetime. However, in our view these excuses do not fully address the arbitrariness of this choice and the partial representation of curvature.

   Note: in our view, the real reason for the choice of Einstein tensor to represent the curvature of spacetime in general relativity is historical and pragmatic since this choice seemed to address and solve certain problems during the formulation of the theory and hence we may describe this as an "empirical" choice (based on a trial and error approach) with no real theoretical justification (although theoretical justifications were improvised later on to rationalize this choice).

6. Comment on the following quote from Synge (see References): "In Einstein's theory, either there is a gravitational field or there is none, according as the Riemann tensor does not or does vanish. This is an absolute property; it has nothing to do with any observer's world-line".

   Answer: This seemingly contradicts our assertion that the Riemann-Christoffel curvature is partially represented in the Field Equation. However, this is not obvious since the equality between the gravitational field and the Riemann-Christoffel curvature could be understood as a controversial claim (i.e. based on the validity of "Einstein's theory"). Moreover, it is in the context of refuting the equivalence principle and hence it is not explicit in stating this equality since the focus in the quote is not this equality. Anyway, this represents Synge's opinion which we are not obliged to follow.

7. It may be claimed that although the curvature of spacetime is not totally represented in the Field Equation and even if the representation by the Einstein tensor is arbitrary, the main empirical criterion for any "correct" theory is its compliance with experiment and observation and hence as long as general relativity is supported by experimental and observational evidence this should not be a defect in the theory. How to address this?

   Answer: First, general relativity is not sufficiently supported by experiment and observation to be "correct" even empirically. In other words, there may still be unverified aspects and consequences related to the missing part of the curvature and the arbitrary choice of representation. Anyway, this does not address the theoretical aspect of this issue since the credibility of the main paradigm of general relativity (i.e. gravity is represented by the curvature of spacetime) should be affected if the curvature is not totally and justifiably represented (regardless of being empirically correct or not). It is obvious that this paradigm has many direct and indirect consequences and hence its credibility is essential to the credibility of the theory in general.

10.1.16 Ambiguity of Vacuum Equation

10.1.17 Need for Classical Gravity

10.1.18 Failure of Geodesic to Replace Force

1. Discuss the issue of geodesic deviation (see § 7.13) in the context of the failure of "geodesic" to replace "force".

   Answer: As we know, the classical paradigms of "gravitational gradient" and "tidal force" do not exist in general relativity (thanks to the geometric approach to gravity and the geodesic paradigm). Although it is commonly claimed that the general relativistic paradigm of "geodesic deviation" replaces these classical paradigms, we believe that geodesic deviation does not replace these paradigms entirely since these paradigms have dynamical content that does not exist in the paradigm of geodesic deviation.[242] Moreover, "gravitational gradient" and "tidal force" have global validity while "geodesic deviation" is essentially local due to the local nature of affine parameter and displacement in the curved spacetime of general relativity. In other words, the formalism of geodesic deviation strictly applies only to neighboring geodesics (which should justify their parameterization by a single natural or affine parameter). In fact, this should also be linked to the issue of global significance of physical quantities in the space-time of classical physics and local significance of physical quantities in the spacetime of general relativity.

2. Assess the paradigms of "geodesic" and "force" from the viewpoint of point-like and extended gravitated objects.
   Answer: The paradigm of "geodesic" is based on point-like objects that trace curves in space and hence it does not lend itself naturally to extended gravitated objects (even if they are assumed to be test objects let alone not to be so). This is unlike the paradigm of "force" which can deal naturally even with extended gravitated objects (although the actual treatment is technically complicated in most cases). In fact, this issue should also be linked to the issue of rigidity in general relativity (see the upcoming note). We should also note that the paradigms of center of gravity and the reduction of an extended gravitated object to a point-like (represented by its center of gravity) should not apply naturally within the general relativistic paradigm of geodesic. In fact, even if we assume that they can apply conceptually and theoretically, there is no evidence that the center of gravity (or its relativistic equivalent) of extended objects should (or will) follow the geodesic trajectory of their point-like equivalent in general.
   Note: extended rigid systems cannot be treated naturally within the framework of general relativity and this should be a handicap and limitation in the theory. In fact, this may explain why extended systems in general relativity are usually modeled as dust or fluid (which are not rigid).

10.1.19 Limitation of Evidence

1. Discuss the issue of the limitation of the claimed evidence in support of general relativity to the gravitational aspects.
   Answer: As explained before, general relativity has two sides: gravity theory and "General Theory". In fact, the validity of general relativity as a "General Theory" depends on the validity of the strong equivalence principle because the strong equivalence principle is what makes general relativity "general relativity" and since there is no evidence in support of the strong equivalence principle (let alone having evidence against it) there is no evidence in support of general relativity as a "General Theory". In brief, all the claimed evidence in support of general relativity is related to the gravitational aspects of the theory and hence even if the evidence is comprehensive and conclusive it should establish general relativity as a gravity theory but no more.
2. What is the significance of the fact that most of the claimed evidence and tests in support of general relativity are related to the Schwarzschild metric in the vacuum region?
   Answer: We note the following:
   • The tests can be considered as tests to the Schwarzschild solution more than to the theory of general relativity in its generality and entirety as represented by the Field Equation.
   • If the Schwarzschild solution (or its physical equivalent in the form of gravitational field for instance) can be obtained from a theory other than general relativity (or from an argument independent of general relativity) then these evidence will essentially lose their significance in endorsing general relativity (at least specifically).
   • Any potential proposition of a different interpretation to the Schwarzschild metric (e.g. as a gravitational field rather than a spacetime attribute) will put a question mark on the necessity (and even correctness) of the curved spacetime paradigm for modeling gravity because this paradigm is the result of the presumed correlation between the spacetime geometry and the physical setting (which is the

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[242] For example, these paradigms are about how and why and not only about how as it is the case with geodesic deviation.

essence of the principle of metric gravity; see § 1.8.4) and this correlation will become questionable if the alleged spacetime geometry can be explained physically by using purely physical concepts and factors.

- The validity of many consequences and implications of the theory cannot be extended automatically to other types of spacetime (i.e. non-Schwarzschild spacetime). In fact, there are many such unjustifiable extensions and generalizations in the literature of general relativity where the results obtained from a particular type of metric (which is usually Schwarzschild) and allegedly validated by these tests are taken as *general relativistic* facts and premises while in reality they are no more than "Schwarzschild facts and premises". In fact, this is a significant limitation of the theory of general relativity itself and it is related to the issues that we discussed earlier about the use of the paradigm of curved spacetime and the proposed correlation between the physics and the curvature since this approach will put metric-restrictions on any obtained result. The reason is that in the case of flat spacetime (e.g. the space-time of classical gravity or the spacetime of Lorentz mechanics) we have a single geometry and hence the obtained results are valid in general regardless of the metric (if the metric is relevant at all) while in the case of curved spacetime we have infinite number of geometries and hence the results (as well as their validation) are metric-specific (see § 2.5.1).
- The restriction of these tests to the vacuum region should also impose more limitations due to the restriction to the vacuum solutions and formalism.

10.1.20 Absurdities and Paradoxes

1. What criteria we use to assess a theory or a premise from the perspective of inconsistency and paradoxes?
   Answer: We mostly use the principles of reality and truth as well as the basic and fundamental rules of logic. We should also note that the existence (or non-existence) of an absolute frame of reference plays a vital role in some of these inconsistencies and paradoxes, and this should highlight and emphasize the importance of this issue in the relativity theories (whether special or general).
2. According to one version of the journey of a freely falling observer A toward a black hole as observed by another observer B at infinity: "B will never see A crossing the event horizon and the image of A will remain frozen at the event horizon for eternity". Comment on this.
   Answer: For example, if A has really entered the event horizon[243] and light cannot come from him then where the light of the eternally frozen image comes from?
   Note 1: for fairness, we should mention that there are more rational versions for describing this journey: e.g. "A gradually slows down and his image is red-shifted until it fades away when A reaches the event horizon". However, almost all versions have some cracks and inconsistencies in their logic. For example, in this improved version we still need a rational explanation for "when A reaches the event horizon" because according to B the time required by A to reach the event horizon is infinite and hence in reality A should never reach the event horizon.
   Note 2: some authors interpret the infinity of the coordinate time (which is equal to the proper time of B) in the above journey as "an external observer [i.e. observer B in our example] never sees the particle [i.e. observer A in our example] cross the horizon". However, this in our view is a confusion between occurrence and observation.
3. Give some examples of general aspects in the theory of general relativity that are common sources of inconsistency and paradoxes.
   Answer: For example:
   - The existence (or non-existence) of an absolute frame of reference for the spacetime of general relativity (see § 6.5).
   - The circularity in some aspects of the theory (see § 10.1.14).
   - The lack of metrical significance of coordinates (see § 6.2, § 7.1 and § 10.1.6).
   - The lack of global significance of many physical quantities in the spacetime of general relativity (see

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[243] Entering the event horizon is either from the perspective of A or because the infinity of the fall time from the perspective of B is apparent (although the latter should be rejected)

§ 7 and § 10.1.8).
- The principle of equivalence (see § 1.8.2 and § 10.1.1).
- The curvature paradigm (and more generally the geometric methodology) as a foundation for gravity (see § 1.8.4).
- The strong non-linearity of the theory.
- The physics of compact objects (although this is an application rather than a general aspect).

4. Give some examples of absurdities and paradoxes that originate from the mathematical formalism of general relativity.
   Answer: There are many examples of this type in the physics of black holes for example (e.g. Schwarzschild and Kerr black holes) where the formalism sometimes leads to traveling back in time or circulation in time or violation of causality. Although these nonsensical aspects of the formalism are usually discarded by general relativists using ready-made standard excuses (such as blaming the coordinates for bad behavior) this is not sufficient to ditch or hide their absurd and paradoxical nature which should affect the rationality and validity of the underlying theory (i.e. general relativity).
5. The blame for paradoxes and absurdities in the applications of general relativity is commonly attached to the inadequacy of the employed coordinate systems (e.g. Schwarzschild coordinates). How do you challenge this?
   Answer: First, this excuse does not change the reality that nonsensical results are obtained from a legitimate application of the theory (even if this is caused by bad coordinates). Moreover, what is the guarantee that the coordinates do not produce similar non-physical (or nonsensical) results without our awareness of their non-physical (or nonsensical) nature? In other words, how do we know that the "good" results of the theory are actually good and not bad results that look good?
6. Give some examples for the general relativistic absurdities from the physics of Schwarzschild black holes.
   Answer: One example is the traveling back in time for the journey of falling object from the event horizon to the singularity (according to the Schwarzschild coordinates).[244] Another example is the prediction of white holes and wormholes (according to the Kruskal-Szekeres coordinates).

10.1.21 The Paradox of Absolute Frame and Reality of Spacetime

10.1.22 Constancy, Invariance and Ultimacy of the Speed of Light

1. What makes the constancy of the speed of light in general relativity local and not global?
   Answer: It is the absence of absolute and global standards for calibrating time and distance due to the dependency of time and length on the coordinates and local metric properties.
2. Discuss the significance of quantum entanglement and its potential impact on the relativity theories.
   Answer: If quantum entanglement and non-local interactions are established (as the current opinion in modern physics seems to be) then all the results and consequences that are based on the presumptions of the existence of a speed limit and the constancy and invariance of the speed of light can be threatened (whether these are found in special relativity or in general relativity). In fact, both theories are generally based (directly or indirectly) on these presumptions and hence thorough inspection and revision of the entire modern physical theories (which are largely based on the validity of the relativity theories and these presumptions in particular) are required where the impact of this profound change should be assessed. We may be forced to go back to some of the old classical views and formulations which do not respect these presumptions, e.g. we may have a gravitational action at a distance similar to the quantum action at a distance. In fact, this could be the most serious threat to the relativity theories in their history so far because it targets some of the most fundamental principles of these theories.
   Note: in our view, the issue of quantum entanglement should have a link to the issue of Mach principle (see § 6.5.1) and the two should have impact on each other. However, this is not of primary interest to the objective and scope of this book and hence we do not go through details.

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[244] The proposed technical fixes (e.g. by swapping t and r) do not fix the apparent absurdity.

10.1.23 Lack of Practicality and Realism

10.1.24 Limited Usefulness

10.2 Overall Assessment

10.2.1 Geometric Nature of General Relativity

10.2.2 High Complexity

10.2.3 Highly Theoretical Nature

10.2.4 Publicizing and Politicizing Science

10.2.5 Controversies, Conflicts and Uncertainties

10.2.6 Practical Value

1. Argue in favor of the redundancy of general relativity practically.
   Answer: From a practical perspective, general relativity is almost redundant since classical gravity is sufficient to deal with most practical situations and the deviation from the Newtonian regime in most general relativistic systems is minimal (as elucidated by Chandrasekhar; see § 9.16). Moreover, most of the characteristic consequences and implications of general relativity can either be obtained from simple arguments and principles (with no need for the full formulation of general relativity) or from alternative theories or from extensions to classical physics. In fact, some can be obtained even empirically. This is unlike quantum mechanics, for example, where it cannot be replaced at the moment by an alternative theory or method and hence it cannot be made redundant.

10.2.7 Theoretical Value

1. What is your overall assessment of general relativity?
   Answer: Our overall assessment of general relativity is that at the best it is a theory of limited value and at the worst it is a theory that hinders the progress of science and can be misleading and even wrong. We should also distinguish between general relativity as a gravity theory and as a "General Theory" where the former seems to be more credible and valuable than the latter.
2. What are the advantages of having intuitive and easy-to-understand scientific theory?
   Answer: We note the following:
   • It is a great advantage for a scientific theory to be intuitive and easy-to-understand because science is a trial to understand the physical world and hence easier and more intuitive science is better in fulfilling the objective of science.
   • Being intuitive and understandable facilitates the detection of any cracks in the theory and helps to avoid errors and mistakes whether theoretical or practical.
   • Intuitive and understandable science is generally more affordable and less expensive. This is because intuitive science is closer to our daily life and experiences and hence it generally requires less sophisticated equipment and methods. Also, being understandable means that less effort and resources are needed to understand the current state of science and develop future scientific theories and procedures.
   • Intuitive science provides a better compass for indicating the future directions and where we should go in our next adventures.
3. Is general relativity ideal?
   Answer: Definitely not, for many reasons some of which are outlined in the text. For example, an ideal scientific theory should be intuitive and this is not the case with general relativity where gravity is described and modeled as a geometric attribute of spacetime. Another example is the highly theoretical nature of this theory (which can lead to illusions and fantasies) and the requirement of sophisticated mathematical machinery (which makes using the theory either impossible or extremely difficult) and this certainly does not serve the progress of science and the use of its resources wisely and efficiently. A third example is its incompatibility with another important theory of physics namely

quantum mechanics and this hinders the unification and progress of science. Therefore, we think it is necessary to search for an alternative to general relativity that addresses those (as well as other) limitations. The insistence on embracing general relativity as the only theory of gravity and hence dismissing any effort to look for an alternative is detrimental to science.

4. Is general relativity correct?
   Answer: There are a number of supporting evidence to this theory although most (or all) of these evidence are not clear cut (refer to § 9). In our view, it is still early to reach a final verdict about its general validity even if we accept all the claimed evidence. For example, most of the claimed evidence are related to certain gravitational regimes or to simple instances where the predictions can be obtained theoretically from alternative theories or from simple arguments with no need for the full formalism of the theory and this should diminish the value of these evidence (see § 10.1.19). Anyway, it seems that the theory contains both correct and incorrect elements where the correct elements are most likely to be in the gravitational aspect of the theory while the incorrect elements are most likely to be in the non-gravitational aspect of the theory. As explained before, almost all the claimed evidence in support of general relativity are related to its state as a gravity theory and not as a "General Theory". Overall, general relativity seems to be neither unconditionally-correct nor unconditionally-incorrect.

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